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Computational Complexity in Analysis

SoSe 2015, Exercise Sheet #12

The lecture defined the *(outer) metric entropy* of a totally bounded metric space (X,d) as the mapping $\lceil X\rceil : \mathbb{N}\to\mathbb{N}$ as follows:
For every $n\in\mathbb{N}$, X can be covered by $2^{\lceil X\rceil(n)}$, but not by $2^{\lceil X\rceil(n)-1}$, open balls of radius 2^{-n}.

The *inner metric entropy* $\lfloor X\rfloor : \mathbb{N}\to\mathbb{N}$ is defined as follows:
For every $n\in\mathbb{N}$, there exist $2^{\lfloor X\rfloor(n)}$, but not $2^{\lfloor X\rfloor(n)+1}$, points of pairwise distance $\geq 2^{-n}$.

For metric spaces (X,d) and (Y,e) and for $L>0$ write

$$\mathrm{Lip}_L(X;Y) \;:=\; \big\{ f:X\to Y \;\big|\; e\big(f(x),f(x')\big)\leq L\cdot d(x,x')\big\}$$

for the set of L–Lipschitz functions. Moreover for $\mu:\mathbb{N}\to\mathbb{N}$ let

$$C_\mu(X) \;:=\; \big\{ f:X\to\mathbb{R} \;\big|\; \forall x,x' : d(x,x')<2^{-\mu(n)} \Rightarrow \big|f(x)-f(x')<\big|\leq 2^{-n} \;\wedge\; |f(x)|\leq 2^{\mu(0)}\big\} \;\; .$$

Finally let $d_S : X\ni x\mapsto \inf\{d(x,s):s\in S\}\in\mathbb{R}\cup\{\infty\}$ denote the distance function of $S\subseteq X$ and $\tilde{d_S} := \min\{1,d_S\}$ its cut-off.

EXERCISE 18:

a) Prove $\lfloor X\rfloor(n)\leq\lceil X\rceil(n)$.

b) Prove $\lceil X\rceil(n)\leq\lfloor X\rfloor(n+1)$.

c) Calculate the metric entropy of $[0;1]$.

d) Calculate the metric entropy of $[0;1]^d$ for every $d\in\mathbb{N}$.

e) Show $d_S\in\mathrm{Lip}_1\big(X,\big[0;2^{\lceil X\rceil(0)}\big]\big)$ for connected X, and $\tilde{d_S}\in\mathrm{Lip}_1\big(X,[0;1]\big)$.

f) Prove $\big\lfloor \mathrm{Lip}_1\big(X,[0;1]\big)\big\rfloor(n)\geq 2^{\lfloor X\rfloor(n)}$. Hint: Take $x_1,\ldots,x_N\in X$ of pairwise distance $\geq 2^{-n}$ and show $\sup_{x\in X}|\tilde{d_S}(x)-\tilde{d_{S'}}(x)|\geq 2^{-n}$ for $S,S'\subseteq\{x_1,\ldots,x_N\}$ with $S\neq S'$.

g) Prove that a set $\mathcal{C}\subseteq C(X)$ is relatively compact iff there exists a $\mu:\mathbb{N}\to\mathbb{N}$ such that $\mathcal{C}\subseteq C_\mu(X)$. Hint: Arzelà-Ascoli.

h) Complement f) by devising a (not necessarily matching) upper bound on $\big\lceil \mathrm{Lip}_1\big(X,[0;1]\big)\big\rceil(n)$.