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CS422 Oct.22, 2015 **Midterm Exam**

Please write your name and student ID here ______________________________

as well as on each additional sheet of paper you use!

35 points = 100%

Problem 1 (8 points): In which of the following cases is the set $L\subseteq\{0,1\}^*$ guaranteed to be *decidable*? Please mark the correct box(es):

☐ If, for every $\underline{x}\in\{0,1\}^*$, there exists an algorithm $\mathcal{A}$ that, on input of $\underline{x}$, answers within a finite number of steps whether $\underline{x}\in L$ or not.

▣ If there exists an algorithm $\mathcal{A}$ that, for every $\underline{x}\in\{0,1\}^*$, on input of $\underline{x}$ answers within a finite number of steps whether $\underline{x}\in L$ or not.

☐ If there exists an algorithm $\mathcal{A}$ that, for some $\underline{x}\in\{0,1\}^*$, on input of $\underline{x}$ answers within a finite number of steps whether $\underline{x}\in L$ or not.

▣ If there exist two algorithms $\mathcal{A}$ and $\mathcal{B}$ where, for every $\underline{x}\in\{0,1\}^*$, $\mathcal{A}$ on input of $\underline{x}$ answers iff $\underline{x}\in L$, and $\mathcal{B}$ on input of $\underline{x}$ answers iff $\underline{x}\notin L$.

☐ If there exists an algorithm $\mathcal{A}$ that ignores its input and prints all $\underline{x}\in L$ in arbitrary order, possibly with repetition.

▣ If there exists an algorithm $\mathcal{A}$ that ignores its input and prints all $\underline{x}\in L$ in lexicographical order without repetition.

▣ If the set L is finite.

▣ If the set L has finite complement $L^C=\{0,1\}^*\setminus L$.

No justification or proofs are required here!

Bonus-Problem (1000 points): Is "no" the only correct answer to this Bonus-Problem?

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Problem 2 (3+3+3 points): Consider the problem of computing, given the coefficients of univariate polynomials $a(x),b(x)\in\mathbb{R}[x]$ of degree $\leq n$, the coefficients of their product $c(x) = a(x) \cdot b(x)$.

a) How many coefficients does the input consist of, how many coefficients are output? How many additions/subtractions/multiplications/division of coefficients does the high-school method use for solving this problem, asymptotically as $n\to\infty$? What asymptotic cost does Karatsuba's Algorithm achieve as discussed in the lecture/homework?

b) Let $a(x) = a_0(x)+x^d\cdot a_1(x)+x^{2d}\cdot a_2(x)$, $b(x) = b_0(x)+x^d\cdot b_1(x)+x^{2d}\cdot b_2(x)$,
and $c(x) = c_0(x)+x^d\cdot c_1(x)+x^{2d}\cdot c_2(x)+x^{3d}\cdot c_3(x)+x^{4d}\cdot c_4(x)$.
Abbreviate $r(x) := \big(4a_0(x)+2a_1(x)+a_2(x)\big) \cdot \big(4b_0(x)+2b_1(x)+b_2(x)\big)$,
$s(x) := \big(a_0(x)+a_1(x)+a_2(x)\big) \cdot \big(b_0(x)+b_1(x)+b_2(x)\big)$,
$t(x) := \big(a_0(x)+2a_1(x)+4a_2(x)\big) \cdot \big(b_0(x)+2b_1(x)+4b_2(x)\big)$.
Verify (2) and (3) of the following:
(1) $c_0(x) = a_0(x)\cdot b_0(x)$
(2) $c_1(x) = -7/2\, a_0(x)\cdot b_0(x) + 1/3\, r(x) - 2\, s(x) + 1/6\, t(x) - a_2(x)\cdot b_2(x)$
(3) $c_2(x) = 7/2\, a_0(x)\cdot b_0(x) - 1/2\, r(x) + 5\, s(x) - 1/2\, t(x) + 7/2\, a_2(x)\cdot b_2(x)$
(4) $c_3(x) = -a_0(x)\cdot b_0(x) + 1/6\, r(x) - 2\, s(x) + 1/3\, t(x) - 7/2\, a_2(x)\cdot b_2(x)$
(5) $c_4(x) = a_2(x)\cdot b_2(x)$

c) Based on (1) to (5) describe a recursive algorithm for the above polynomial multiplication problem that improves over Karatsuba. What asymptotic cost does it achieve?

Problem 3 (3+3+3 points): Formalize the following decision problems as subsets of $\{0,1\}^*$. Which of them are decidable? Prove your answers, either by describing an algorithm or by establishing a computable reduction from the Halting problem.

a) Given a finite sequence of brackets "`(`" and "`)`", are they correctly nested?

b) Given an algorithm $\mathcal{B}$, is the set of inputs $\underline{x}$ on which $\mathcal{B}$ terminates finite?

c) Given a finite automaton $\mathcal{A}$, does there exist a finite binary string $\underline{x}$ that $\mathcal{A}$ accepts?

Problem 4 (3+3+3 points): Recall the bijection $\mathbb{N}^2 \ni (x,y) \to \langle x,y\rangle := 2^x\cdot(2y+1)-1 \in \mathbb{N}$.

a) Devise a LOOP program that, given x and y, computes $\langle x,y\rangle$.

b) Devise a LOOP program that, given $\langle x,y\rangle$, computes x and y.

c) We define $\langle x,y,z\rangle := \langle\langle x,y\rangle,z\rangle$, $\langle x,y,z,w\rangle:=\langle\langle x,y,z\rangle,w\rangle$, and so on inductively.
Devise a LOOP program that, given integers $k\leq n$ and $\langle x_1,x_2,\ldots,x_k\ldots,x_n\rangle$, returns x_k.