# §1 Stable Matching



## Motivation: Matching KAIST students with labs <u>automatically</u> (algorithm!) to find <u>stable</u> solution.


**Inputs: a)** each student's order of preferred labs
**b)** each lab's order of preferred students

**Output:** 1-1 pairing w/out *unstable* tuples

**Def:** Tuple $(S,P)$ is *unstable* if $S$ prefers $P$ over assigned $P'$ and $P$ prefers $S$ over assigned $S'$

# Stable Matching

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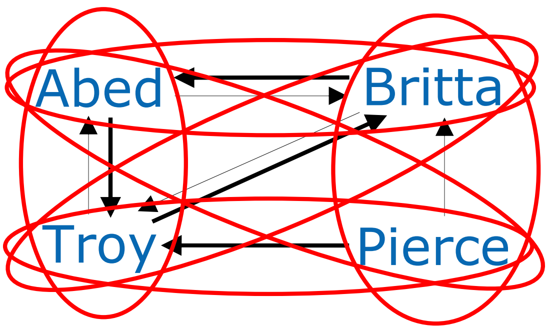
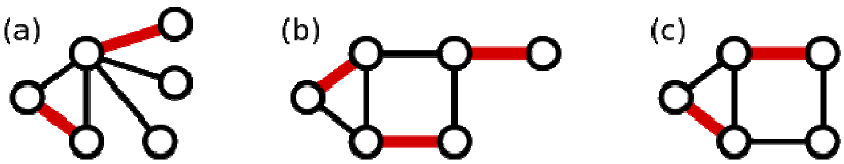
## Does it always exist? No!




**Reminder:** *A perfect* matching in a graph $G=(V,E)$ of $|V|=2n$ vertices is a subset $M$ of $n$ edges without common vertices.




**<u>Specification:</u>**

**Input:** $n$ 'men' and $n$ 'women', each with a ranking of preference among the opposite 'gender'.

**Output:** stable perfect matching

**Def:** Tuple $(w,m)$ is *unstable* if $w$ prefers $m$ over assigned $m'$ and $m$ prefers $w$ over assigned $w'$

# Stable Matching Algorithm



Gale-Shapley (1962)

$M$ := {}

WHILE some $m$ is unmatched

  Let $m$ propose to $w$ := first on $m$'s list that $m$ has not yet proposed to.

  IF $w$ is unmatched, add $(m,w)$ to $M$

  ELIF $w$ prefers $m$ to current partner $m'$ replace $(m',w)$ in $M$ with $(m,w)$

  ELSE $w$ rejects proposal from $m$.

ENDWHILE // output: $M$

**Specification:**

**Input:** $n$ 'men' and $n$ 'women', each with a ranking of preference among the opposite 'gender'.

**Output:** 'matching' w/out *unstable* tuples

**Def:** Tuple $(w,m)$ is *unstable* if $w$ prefers $m$ over assigned $m'$ and $m$ prefers $w$ over assigned $w'$

# Proof of Correctness

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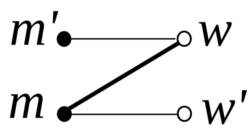
**Observation A:** Once a woman is matched, she never becomes unmatched but only "trades up".




**Observation B:** Any man proposes to women in decreasing order of preference.

$M$ := {}

WHILE some $m$ is unmatched

  Let $m$ propose to $w$ := first on $m$'s list that $m$ has not yet proposed to.

  IF $w$ is unmatched, add $(m,w)$ to $M$

  ELIF $w$ prefers $m$ to current partner $m'$ replace $(m',w)$ in $M$ with $(m,w)$

  ELSE $w$ rejects proposal from $m$.

ENDWHILE // output: $M$

**Claim 1:** The loop terminates after $\leq n^2$ iterations.

**Claim 2:** All get matched.

**Claim 3:** Matching w/o unstable pairs.

**Def:** Tuple $(w,m)$ is *unstable* if $w$ prefers $m$ over assigned $m'$ and $m$ prefers $w$ over assigned $w'$

# Efficiency: implement in $O(n^2)$



Represent men by numbers $1\ldots n$; same for women.

**Input:** $n$-element arrays with order of preference for each $m,w=1\ldots n$

**Output:** matching, represented by two $n$-element arrays wife$[m]=w$ and husband$[w]=m$; $=0$ if unmatched.

WHILE some $m$ is unmatched

  Let $m$ propose to $w$ := first on $m$'s list that $m$ has not yet proposed to.

  IF $w$ is unmatched, add $(m,w)$ to $M$

  ELIF $w$ prefers $m$ to current partner $m'$ replace $(m',w)$ in $M$ with $(m,w)$

  ELSE $w$ rejects proposal from $m$.

ENDWHILE // output: $M$

For each man $m$, lastwproposed$[m]$

For each woman $w$, inverted order of preference.

**Is this running time optimal?**

# Understanding the Solution



Represent men by numbers $1\ldots n$; same for women.

**Input:** $n$-element arrays with order of preference for each $m,w=1\ldots n$

**Example** [two stable matchings]

| | 1st | 2nd | 3rd |
|---|---|---|---|
| Abed | Annie | Britta | Frankie |
| Ben | Britta | Annie | Frankie |
| Craig | Annie | Britta | Frankie |

| | 1st | 2nd | 3rd |
|---|---|---|---|
| Annie | Ben | Abed | Craig |
| Britta | Abed | Ben | Craig |
| Frankie | Abed | Ben | Craig |

{ (Abed,Annie) , (Ben,Britta) , (Craig,Frankie) }

{ (Abed,Britta) , (Ben,Annie) , (Craig,Frankie) }

Gale-Shapley produces *that* stable matching why? where every $m$ gets assigned his *most* preferred choice among all $w$ matched to him in *any* stable matching; whereas $w$ gets assigned her *least* preferred choice.