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CS700 Oct.20, 2016 Midterm Exam

Please write your name and student ID here ________________________________

as well as on each additional sheet of paper you use!

50 points = 125%

Problem 1 (3+3+4 points):

a) Let $A\subseteq\mathbb{N}$ be decidable and $\emptyset\subset B\subset\mathbb{N}$ arbitrary. Prove $A\preceq B$.

b) Prove $H\preceq T$, where H denotes the Halting Problem and T the Totality Problem:

$H = \{ \langle\mathcal{A},x\rangle : \mathcal{A}$ terminates on input $x \}$, $T = \{ \langle\mathcal{A}\rangle : \mathcal{A}$ terminates on all inputs $\}$

c) Prove $T\not\preceq\overline{H}$.

Recall that a real number r is computable iff some (equivalently: all) of the following hold:

a) r has a decidable binary expansion

b) There exists a computable integer sequence (a_m) s.t. $\forall m$: $|r-a_m/2^m|\leq 2^{-m}$.

c) There exist computable sequences (q_n) and (ε_n) of (numerators and denominators of) rational numbers such that $|r-q_n|\leq\varepsilon_n\to 0$.

Problem 2 (3+4+3 points):

a) Give an example of a real number r which is *not* computable.

b) Conclude that this number is transcendental.

c) Prove that every non-empty open interval contains a computable real.

Problem 3 (3+4+3 points):

a) Let a,b be computable real numbers. Prove that $a+b$ is computable.

b) Let a,b be computable real numbers. Prove that $a\cdot b$ is computable.

c) Let (a_j), (b_j) be computable sequences. Prove that (a_j+b_j) is computable.

Recall that a real sequence (r_j) is called computable iff there exists a computable integer double sequence $a_{j,m}$ such that $\forall m,j$: $|r_j-a_{j,m}/2^m|\leq 2^{-m}$.

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Recall that a real function f:[0;1]→ℝ is <u>computable</u> iff some Turing machine can convert any integer sequence a_m satisfying $|x-a_m| \leq 2^{-m}$ for some x, into an integer sequence b_n s.t. $|f(x)-b_n| \leq 2^{-n}$; equivalently: there exists a computable sequence of (degrees and coefficient lists of) integer polynomials p_m such that $\|f-p_m\|_\infty \leq 2^{-m}$.

Problem 4 (3+ 4+ 3 points):

a) Let $(r_j)\subseteq[0;1]$ denote a computable sequence of real numbers and f:[0;1]→ℝ a computable real function. Prove that $(f(r_j))$ constitutes again a computable real sequence.

b) Let (r_j) denote a computable sequence of real numbers such that $|r_j-r_k| \leq 2^{-j}+2^{-k}$. Prove that $r:=\lim_j r_j$ exists and is a computable real number.

c) Give an example of a computable real sequence (r_j) in [0;1] which does *not* have a computable accumulation point.

Problem 5 (10 points): Let (r_j) denote an arbitrary computable sequence of real numbers. Without proofs, check (√) which of the following sets are

	$\{ j\in\mathbb{N} : r_j = 0 \}$	$\{ j\in\mathbb{N} : r_j \neq 0 \}$	$\{ j\in\mathbb{N} : r_j > 0 \}$	$\{ j\in\mathbb{N} : r_j \geq 0 \}$
decidable				
semi-decidable				
co-semi-decidable				
recursively enumerable				