Revisiting and Tailoring Auction Theory for Eurex Clearing

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in cooperation with

Eurex Clearing AG - Risk Design

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Frankfurt am Main, February 15, 2012

To my wife Samiye

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig angefertigt habe. Sämtliche aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht.

Die Arbeit wurde bisher keiner anderen Prüfungsbehörde vorgelegt und noch nicht veröffentlicht.

Frankfurt am Main, den 15. Februar 2012 Selim Dogruel

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Contents

List of Figures

1 Introduction

1.1 What is a clearing house / CCP?

A clearing house is a financial institution that provides a full range of different services for financial transactions. A clearing house can also act as the central counterparty (CCP) for its customers. Instead of bilateral trades the CCP acts as the seller to every buyer and the buyer to every seller. Here, there are two possible ways the clearing house can become the CCP. The first concept is *open offer* where the CCP automatically becomes the counterparty for all transactions at the supported exchange markets. The second concept is *novation*, through this process "a bilateral contract between two market participants is replaced by two bilateral contracts between each of the original counterparties and the CCP."[1]

1.2 Member structure at Eurex Clearing

Eurex is a derivatives exchange run by the Group Deutsche Boerse AG. A variety of different products are offered to members from 700 locations worldwide, who generate a trading volume of over 1.5 billion contracts a year. Eurex Clearing is the clearing house of Eurex and acts as a central counterparty.[2] In general, Eurex Clearing offers three types of memberships to potential new customers.[3] *General Clearing Member* (GCM) are able to clear for themselves and for their customers, aswell as for associated *Non Clearing Member* (NCM), which form another group of members. *Direct Clearing Member* (DCM) are the last type of members. They are only allowed to clear for themselves, their customers and affiliated NCMs.

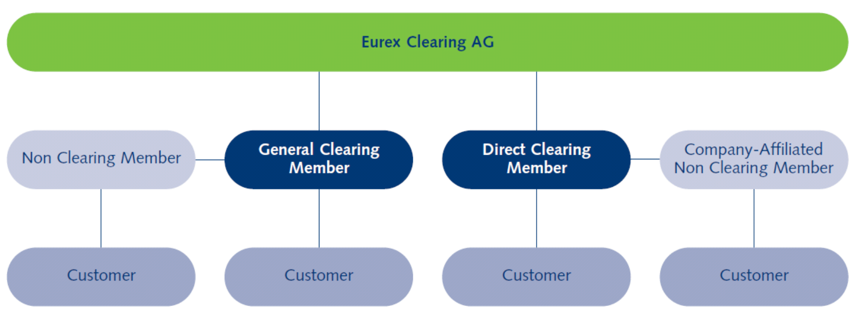

Figure 1.1: Member Structure of Eurex Clearing[4]

[1]Bliss, R. and Papathanassiou, C.: Derivatives clearing, central counterparties and novation: The economic implications, p. 3.

[2]Eurex Clearing also clears other markets in addition to Eurex (e.g. the equity trading on the Frankfurt Stock Exchange/Xetra). However, Eurex Derivatives are of interest in this thesis and these products will be described.

[3]For each membership there are different obligations and requirements.

[4]Source: Eurex Clearing AG: Risk Based Margining, p. 8.

1.3 Margin process and lines of defense

In order to be able to offset each open position in the portfolio within a very short period of time, the clearing member has to deposit securities as collateral, which is called *margin*. There are different types of margins. While "premium, current liquidating and variation margin cover the liquidation risk at the current point of time, additional and future spread margins cover the potential liquidation risk for the following business day." The advantage for the members in paying margins lies in the fact that they only have to provide coverage for the risk of their positions instead of paying the full value of the open positions. A further advantage is that Eurex Clearing takes risk reducing effects of different positions into account. Thus, "equal but opposite risks within the account are offset against each other", which yields to a lower amount of collateral the clearing member has to provide.[5]

Since a CCP must not default even in extreme market situations, Eurex Clearing has established multiple lines of defense to be protected in case of a default of a clearing member (CM). Beside the obligation to provide margins, CMs have to provide a contribution to the Clearing Fund. This security system should ensure that Eurex Clearing can "guarantee the fulfillment of every contract on every market where they provide their services."[6] In figure 1.2, the different lines of defense of Eurex Clearing are illustrated.

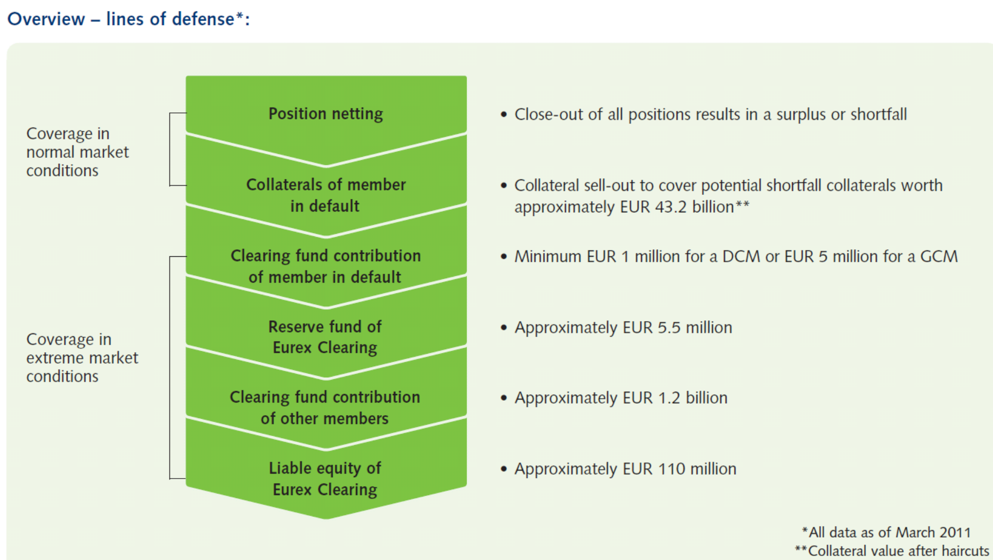

Figure 1.2: Eurex Clearing's lines of defense[7]

[5] http://www.eurexclearing.com/risk/margin_process_en.html
[6] http://www.eurexclearing.com/risk/line_defense_en.html
[7] Source: Eurex Clearing AG: Safeguards of the clearing house, p. 16.

1.4 Overview on derivatives

Derivatives are financial instruments. Their value depends on some other financial product, the so called underlying (i.e. the value is *derived* from this underlying).[8]

1.4.1 Futures

Futures are standardized contracts between two parties which are traded on an exchange. Basically, a future is an agreement to buy or sell a financial product at a specific time for a specific price. One of the two parties is in a *long position* and has to buy the underlying at the specific time for the specific price. The other party is in a *short position* and has to sell the respective product at the same time for the same price. Settlement takes place at maturity. The party in the short position delivers the underlying product to the party in the long position and receives in return the payments of the agreed price (*strike price*).[9]

1.4.2 Options

Options are financial contracts which give, unlike futures, the right (and not the obligation) to buy or sell a certain financial product (*the underlying*) at a specific time (*maturity*) for the predetermined strike price. While a *call option* gives the buyer of the option the right to buy the underlying, a *put option* gives the buyer the right to sell the underlying asset. Again, the buyer of an option is in a long position and the seller is in a short position. One distinguishes between american and european options. American options can be exercised at any time until maturity but european options can only be exercised at maturity date.[10]

1.5 Default of a clearing member

A clearing member defaults when it has not fulfilled its legal obligations. In this context, insolvency or bankruptcy are only a stronger form of a default. But a default of a CM can also occur when it does not pay its margins. Even when there is a delay in the payments Eurex Clearing can declare a technical default of the CM. When a clearing member default occurs, Eurex Clearing has to transfer its risk exposure resulting from such a default to another Clearing Member. Eventual losses in this process are covered by the lines of defense. If the resources of the lines of defense are not sufficient to cover the losses, Eurex Clearing "has the right to request the non-defaulting clearing members to replenish the clearing fund."[11]

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[8]See: Hull, J. C.: Options, Futures and Other Derivative Securities, p. 1.
[9]See: Hull, J. C.: Options, Futures and Other Derivative Securities, pp. 2-3.
[10]See: Kohler, J.: Einfuehrung in die Finanzmathematik, p. 3.
[11]International Monetary Fund: Country Report No. 11/271, p. 20.

In the figure below we can see a situation before the default of a clearing member.

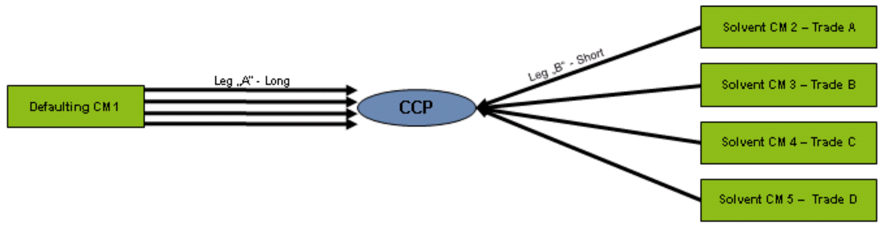

Figure 1.3: CCP is balanced before CM default[12]

If now in the example below CM1 defaults, the CCP is not balanced anymore. Eurex Clearing as the CCP becomes the legal owner of the positions and the margin collateral.

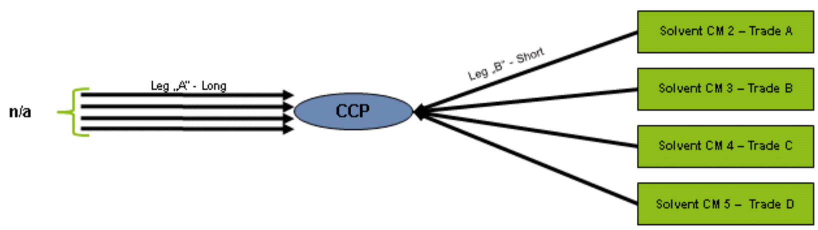

Figure 1.4: Default of CM1, CCP is unbalanced[13]

In a next step Eurex Clearing "will re-establish opposing transactions with the market so that the risk exposure for the CCP is flattened."[14]

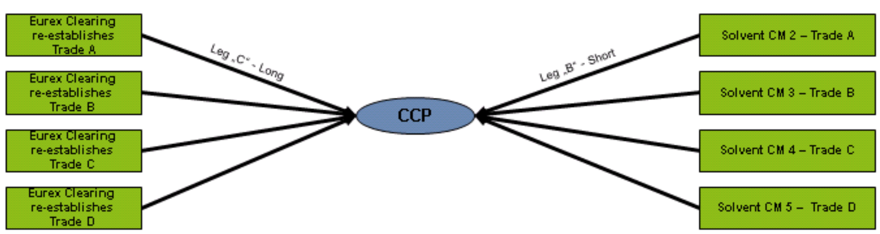

Figure 1.5: Re-balanced CCP[15]

As the case of the insolvency of Lehman Brothers has shown, a default of a (major) clearing member has a lot of negative effects on the whole market.

[12] Source: Eurex Clearing AG: Detailed Liquidation Scenario Document.
[13] Source: Eurex Clearing AG: Detailed Liquidation Scenario Document.
[14] Eurex Clearing AG: Detailed Liquidation Scenario Document, p. 11.
[15] Source: Eurex Clearing AG: Detailed Liquidation Scenario Document.

Thus, Eurex Clearing strives for a default management process minimizing the negative effects of a default.

1.6 Proposals on the new default management process

Eurex Clearing will introduce a new default management process in which auctions will play the key role. In case of a default of a clearing member, the portfolio will be divided in so-called "liquidation-groups". After a hedging process[16] the different liquidation groups will then be auctioned off separately.
The new process has the following key goals:

1. "Minimize the effect and disruption on the membership and the wider market.

2. Minimize losses to the clearing house's lines of defense i.e. minimize losses to contributions from the clearing member community and the clearing house."[17]

More precisely, the following steps are considered to liquidate efficiently the portfolio of the defaulted clearing member.[18]

1. *Predefined Liquidation Groups:* In order to facilitate further steps such as hedging and pricing, all products of the portfolio will be split into parts, so called liquidation groups.

2. *Preliminary Actions:* Since there may be positions in the portfolio with approaching maturity, preliminary actions can be performed to minimize or close-off such risks.

3. *Hedging:* Before the liquidation groups will be sold via independent sale or auctions, an asset-class specific hedging will take place. The reason for the hedging is to make the portfolio "less sensitive to market moves". Furthermore, the hedged liquidation group becomes more stable in the auction, since "bidders' future expectations have less impact on the portfolio price."

4. *Independent Sale and Auction:* The CCP may liquidate small portfolios or parts of a portfolio with special products via bilateral trades or by order book trading. But, as stated above, auctions are the core component of the new procedures in order to establish a fair market price. Thus, a separate auction will be established for each liquidation group.

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[16]Hedging describes all steps with the goal to minimize possible volatilities in the price of a portfolio, for example by buying a financial instrument which can offset the risk of the existing positions. See: Oil and Gas Investor, White Paper: Hedging Commodity Risk, Allegro Comodity Corp, 2009, p. 1.

[17]Eurex Clearing AG: Project Prisma, Default Management Process.

[18]Note that positions and collateral of the clients of the defaulted clearing member will be transfered to other clearing members as a first step.

5. *Allocation:* In case that all other attempts to liquidate parts of the portfolio did not succeed, Eurex Clearing allocates the remaining parts to its members "in order to offset the risk exposure for the clearing house."[19]

The proposed process for the auctions can be described as follows:

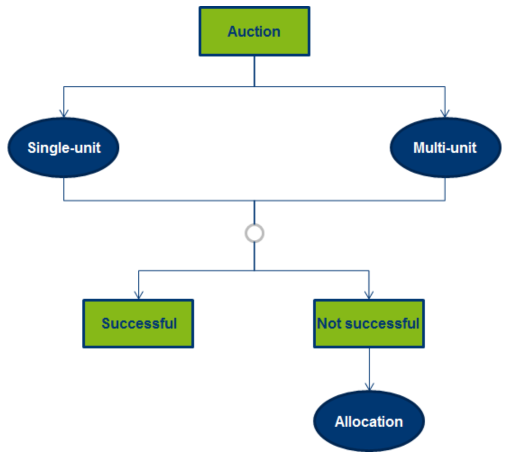

Figure 1.6: Auction Scheme

For each hedged liquidation group, a separate auction will be set up, generally there are no restrictions for participation in the auctions. Thus each clearing member (and also their clients) is allowed to take part in the auctions. There are two possibilities for the setup of the auction. The first possibility is a single-unit auction, i.e. the whole liquidation group will be auctioned off. The winner of the auction will receive the liquidation group and has to pay the price depending on the chosen auction format.[20] The second possibility is a multi-unit auction. This means that the respective liquidation group will be split in homogenous parts and bidders are able to bid on portions instead of the whole liquidation group. If the auction is not successful (i.e. no competitive price for the liquidation group or parts of it was established), Eurex Clearing can directly allocate parts of the portfolio.[21]

1.7 Why are auctions useful?

Let us first think about the question what an auction actually is and what kind of processes fall under this category. What all different auction formats have in common is the fact that they extract information by the potential buyers of an object to be sold (via the bidding process). Two further aspects are

[19] Eurex Clearing AG: Detailed Liquidation Scenario Document.

[20] Please note: Since financial products/derivatives are being auctioned off bids with negative values are also possible. In this case, the winner of the auction receives the liquidation group and the respective price.

[21] Eurex Clearing AG: Detailed Liquidation Scenario Document.

also characteristic for auctions. These are the determination of the winner and the determination of the payments by the auction participants. The determination processes are only based on the received information, i.e. auctions are anonymous, the identity of the potential buyers does not influence the determinations.[22]

Today, a variety of auctions are used in many different areas. For example, the U.S. Treasury sells long-term securities via a weekly auction, in Aalsmeer (Netherlands) the worlds' biggest flower auction takes place every day. Also in the case of the default of Lehman Brothers in 2008, auctions were used to liquidate the portfolio of Lehman Brothers, but in a very unorganized way. But an organized auction in case of a default would generate more liquidity and reduce price disruptions,[23] "than uncoordinated replacement of positions during periods of pronounced uncertainty."[24] Also Craig Pirrong[25] considers auctions as a good strategy for a CCP to "reduce the disruptive effects of default."[26] Regarding the questions why auctions are used, Vijay Krishna gives the following answer: "Auctions are used precisely because the seller is unsure about the values that bidders attach to the object being sold."[27] This means that if a seller would know the values of the potential buyers, he would just offer the object to the potential buyer with the highest value.

1.8 Motivation

Plenty of excellent work exists on auction theory dealing with the design of auctions, the right incentive management and results and backgrounds on efficiency and revenue considerations of different auction formats. Since existing literature on auction theory adresses mainly readers from the economy, the mathematical aspects and details are kept very functional, definitions and theorems are designed in a way to serve only the predefined settings.
This thesis will review auction theory in a unified mathematical framework. The focus will lie on the analysis of the maybe most celebrated theorem of auction theory: The revenue equivalence theorem. The development of this theorem will be characterized and a more general version will be presented for the single-unit and for the multi-unit version. There will be an outlook on further aspects of auction theory and the implications will be assessed, leading to recommendations for suitable central counterparty procedures at Eurex Clearing.

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[22]See: Krishna, V.: Auction Theory, Academic Press, 2009, p. 6.
[23]See: Greenwald, B. and Stein, J.: Transactional Risk, Market Crashed and the Role of Circuit Breakers 64 J. of Business (1991)443.
[24]Pirrong, C.: ISDA Discussion Paper Series, The Economics of Central Clearing: Theory and Practice, p. 11.
[25]Craig Pirrong is Professor of Finance, and Energy Markets Director for the Global Energy Management Institute at the Bauer College of Business at the University of Houston.
[26]Pirrong, C.: ISDA Discussion Paper Series, The Economics of Central Clearing: Theory and Practice, p. 2.
[27]Krishna, V.: Auction Theory, Academic Press, 2009, p. 2.

2 Introduction to auction theory

2.1 General framework

In this section we want to begin to define a general framework and set the notations in order to present and analyse the theorems in the next chapters. First, let us think about what is neccessary in order to perform an auction. In fact, each auction consists of two components, an *allocation rule*, determining who will get the object(s) auctioned-off by the seller and a *payment rule*, specifying the payments after the auction by each participant.

Definition 2.1 (Single-Unit Auction). *Let n be a natural number and let A_0 be a function defined as follows:*

$$\begin{array}{llll} A_0 : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & q^A(x) \end{array}$$

$$\begin{array}{l} \textit{where } q^A(x) = \big(q^A(x)_1, \dots, q^A(x)_n\big), \\ \forall i \in \{1, \dots, n\}, 0 \leq q^A(x)_i \textit{ and } \sum_{i=1}^{n} q^X(x)_i \leq 1. \end{array}$$

Furthermore, let $A_1, \dots, A_n$ be functions of type $\mathbb{R}^n \to \mathbb{R}$. Then, a single-unit auction A is defined by $A = (A_0; A_1, \dots, A_n)$.

Definition 2.2 (Multi-Unit Auction). *Let n and k be natural numbers and let A_0 be a function defined as follows:*

$$\begin{array}{llll} A_0 : & (\mathbb{R}^k)^n & \to & (\mathbb{R}^k)^n \\ & x & \mapsto & q^A(x) \end{array}$$

$$\begin{array}{l} \textit{where } q^A(x) = \big(q^A(x)_1, \dots, q^A(x)_n\big), \\ \forall i \in \{1, \dots, n\}, q^A(x)_i = (q_1^A(x)_i, \dots, q_k^A(x)_i), \\ \forall l \in \{1, \dots, k\}, q_l^A(x) = \big(q_l^A(x)_1, \dots, q_l^A(x)_n\big) \\ \textit{and } \forall i \in \{1, \dots, n\} \textit{ and } \forall l \in \{1, \dots, k\}, q_l^A(x)_i \in [0, 1]. \end{array}$$

Furthermore, let $A_1, \dots, A_n$ be functions of type $(\mathbb{R}^k)^n \to \mathbb{R}$. Then, a multi-unit auction A is defined by $A = (A_0; A_1, \dots, A_n)$.

Beside an allocation and a payment rule, we need to define bidding functions.

Definition 2.3 (Bidding function). *Let k be natural number and let b be a function of type $\mathbb{R}^k \to \mathbb{R}^k$.*

In the next two subchapters we present some standard auction formats, single-unit and multi-unit. Here we will use the formalism above, i.e. an allocation and a payment function per bidder with corresponding bidding functions.

2.2 Single-unit auctions

Definition 2.4 (First price auction). *Let n be a natural number. For all i between 1 and n, let v_i be a real number. For all $i \in \{1, \dots, n\}$, let the functions*

b_i be of type $\mathbb{R} \to \mathbb{R}$ and let the functions F_0 and F_i be defined as follows.

$$\begin{array}{llll} F_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & v & \mapsto & q^F(v) \end{array}$$

$$\begin{aligned} &\textit{where } q^F(v) = \big(q^F(v)_1, \ldots, q^F(v)_n\big) \\ &\textit{and } \forall i \in \{1, \ldots, n\}, q^F(v)_i = \begin{cases} 1 & \textit{if} \quad b_i(v_i) > \max_{j \neq i} b_j(v_j) \\ 0 & \textit{if} \quad b_i(v_i) < \max_{j \neq i} b_j(v_j) \end{cases} \end{aligned}$$

$$\begin{array}{llll} F_i: & \mathbb{R}^n & \to & \mathbb{R} \\ & v & \mapsto & \begin{cases} b_i(v_i) & \textit{if} \quad b_i(v_i) > \max_{j \neq i} b_j(v_j) \\ 0 & \textit{if} \quad b_i(v_i) < \max_{j \neq i} b_j(v_j) \end{cases} \end{array}$$

$$\textit{where } v = (v_1, \ldots, v_n)$$

In a first price auction each bidder $i \in \{1, \ldots, n\}$ has a value $v_i \in \mathbb{R}$ for the object to be auctioned-off. Each bidder submits a bid $b_i(v_i)$ and he gets the object if and only if his bid is the highest among all bids. In this case he gets the object and has to pay an amount equal to his submitted bid. All other bidders do not have to pay anything. In case of a draw (two or more bidders submitted the highest bid), some arbitrary rule may be chosen to determine the winner of the auction.

Definition 2.5 (Second price auction)**.** *Let n be a natural number. For all i between 1 and n, let v_i be a real number. For all $i \in \{1, \ldots, n\}$, let the function b_i be of type $\mathbb{R} \to \mathbb{R}$ and let the functions S_0 and S_i be defined as follows.*

$$\begin{array}{llll} S_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & v & \mapsto & q^S(v) \end{array}$$

$$\begin{aligned} &\textit{where } q^S(v) = \big(q^S(v)_1, \ldots, q^S(v)_n\big) \\ &\textit{and } \forall i \in \{1, \ldots, n\}, q^S(v)_i = \begin{cases} 1 & \textit{if} \quad b_i(v_i) > \max_{j \neq i} b_j(v_j) \\ 0 & \textit{if} \quad b_i(v_i) < \max_{j \neq i} b_j(v_j) \end{cases} \end{aligned}$$

$$\begin{array}{llll} S_i: & \mathbb{R}^n & \to & \mathbb{R} \\ & v & \mapsto & \begin{cases} max_{j \neq i} b_j(v_j) & \textit{if} \quad b_i(v_i) > \max_{j \neq i} b_j(v_j) \\ 0 & \textit{if} \quad b_i(v_i) < \max_{j \neq i} b_j(v_j) \end{cases} \end{array}$$

$$\textit{where } v = (v_1, \ldots, v_n)$$

In the second price auction, there are again $n \in \mathbb{N}$ bidders competing for a single object. As in the first price auction, the bidder with the highest bid wins the auction and gets the object. But in contrast to the first price auction, the winner of the auction does not pay his own bid but instead the second highest bid. The two single-unit auctions we defined so far have one property in common: They are both sealed bid auctions, i.e. the bidding process is secret, only the seller gets to know how everyone has bid. Beside these two common auction formats, there are also two well-known "open" auction formats, which we want to quickly describe.

Definition 2.6 (Dutch Auction)**.** *The seller publicly announces a price for the object to be auctioned off. This price is lowered step by step until one bidder indicates that he wants to buy the object (for example by pressing a button). This bidder gets the object and has to pay this respective price.*

Definition 2.7 (Progressive (English) Auction)**.** *The seller publicly announces a price for the object to be auctioned off. Each bidder who is interested in buying the object for this price indicates his interest (for example by pressing a button). The seller raises the price step by step as long as there are at least two bidders remaining with the intend to buy the object. The auction ends when there is only one bidder left. He gets the object and pays the price at which the second-last bidder drops out of the bidding process.*

There are some relations between the "sealed" and "open" auctions. For example, we notice that the dutch auction is strategically equivalent to the first price auction. Strategical equivalence means, that "for every strategy in one [auction], a [bidder] has a strategy in the other [auction], which results in the same outcomes."[28] Although we mentioned that the Dutch auction belongs to the "open" auction formats, there is no useful information for the bidders. The only time when the bidders get information is when one of them agrees to buy the object, but at this point the auction terminates. So making a bid $b(v)$ in the first price auction is equivalent to signalling the willingness to buy the object for $b(v)$ in the dutch auction (of course only when the object has not been sold yet)[29].

2.3 Multi-unit auctions

Definition 2.8 (Discriminatory auction)**.** *Let n and k be natural numbers. For all i between 1 and n, let $v_i \in \mathbb{R}^k$. For all $i \in \{1, \ldots, n\}$, let the function b_i be of type $\mathbb{R}^k \to \mathbb{R}^k$ and let $b_i(v_i) = (b_i^1(v_i^1), \ldots, b_i^k(v_i^k))$. For all $i \in \{1, \ldots, n\}$ and for all $l \in \{1, \ldots, k\}$, let C be a set containing the k highest of all $b_i^l(v_i)$.*

[28] Krishna, V.: Auction Theory, Academic Press, 2009, p. 4.
[29] See: Krishna, V.: Auction Theory, Academic Press, 2009, p. 4.

For all i between 1 and n, let the functions D_0 and D_i be defined as follows.

$$\begin{array}{llll} D_0: & (\mathbb{R}^k)^n & \to & (\mathbb{R}^k)^n \\ & v & \mapsto & q^D(v) \end{array}$$

$$\begin{aligned} &\textit{where } q^D(v) = \big(q^D(v)_1, \ldots, q^D(v)_n\big), \\ &\forall l \in \{1, \ldots, k\}, q_l^D(v) = \big(q_l^D(v)_1, \ldots, q_l^D(v)_n\big) \\ &\textit{and } q_l^D(v)_i = \begin{cases} 1 & \textit{if} \quad b_i^l(v_i^l) \in C \\ 0 & \textit{if} \quad b_i^l(v_i^l) \notin C \end{cases} \qquad \forall l \in \{1, \ldots, k\} \end{aligned}$$

$$\begin{array}{llll} D_i: & (\mathbb{R}^k)^n & \to & \mathbb{R} \\ & v & \mapsto & \sum_{b_i^l(v_i^l) \in C} b_i^l(v_i^l) \end{array}$$

$$\textit{where } v = (v_1, \ldots, v_n)$$

In multi-unit auctions we have $n \in \mathbb{N}$ bidders competing for $k \in \mathbb{N}$ identical objects. Thus, each bidder i has a valuation $v_i = (v_i^1, \ldots, v_i^k)$ and submits k bids $b_i^l(v_i^l)$, with $b_i^1(v_i^1)$ representing bidder i's bid for the first object. In the discriminatory auction a bidder wins the l-th object if his l-th bid is in the set C. For each object the bidder wins he pays an amount equal to his successful bid.

Definition 2.9 (Uniform price auction)**.** *Let n and k be natural numbers. For all i between 1 and n, let $v_i \in \mathbb{R}^k$. For all $i \in \{1, \ldots, n\}$, let the function b_i be of type $\mathbb{R}^k \to \mathbb{R}^k$ and let $b_i(v_i) = (b_i^1(v_i^1), \ldots, b_i^k(v_i^k))$. For all $i \in \{1, \ldots, n\}$ and for all $l \in \{1, \ldots, k\}$, let C be a set containing the k highest of all $b_i^l(v_i^l)$. Furthermore, let $h \in \mathbb{R}$ identify the $k+1$st value of all $b_i^l(v_i^l)$. For all i between 1 and n, let $t_i := \#(b_i^l(v_i^l) \in C)$ and let the functions U_0 and U_i be defined as follows.*

$$\begin{array}{llll} U_0: & (\mathbb{R}^k)^n & \to & (\mathbb{R}^k)^n \\ & v & \mapsto & q^U(v) \end{array}$$

$$\begin{aligned} &\textit{where } q^U(v) = \big(q^U(v)_1, \ldots, q^U(v)_n\big), \\ &\forall l \in \{1, \ldots, k\}, q_l^U(v) = \big(q_l^U(v)_1, \ldots, q_l^U(v)_n\big) \\ &\textit{and } q_l^U(v)_i = \begin{cases} 1 & \textit{if} \quad b_i^l(v_i^l) \in C \\ 0 & \textit{if} \quad b_i^l(v_i^l) \notin C \end{cases} \qquad \forall l \in \{1, \ldots, k\} \end{aligned}$$

$$\begin{array}{llll} U_i: & (\mathbb{R}^k)^n & \to & \mathbb{R} \\ & v & \mapsto & t_i \cdot h \end{array}$$

$$\textit{where } v = (v_1, \ldots, v_n)$$

In the uniform price auction the winning bids are again those bids which are among the k highest submitted bids. But in constrast to the discriminatory auction the payment rule is as follows: First the highest bid $h \in \mathbb{R}$ which is not among the k winning bids will be identified (i.e. h is the highest losing bid). All winning bidders now have to pay this price h for each successful bid.

Definition 2.10 (Vickrey auction)**.** *Let n and k be natural numbers. For all i between 1 and n, let $v_i \in \mathbb{R}^k$. For all $i \in \{1, \dots, n\}$, let the function b_i be of type $\mathbb{R}^k \to \mathbb{R}^k$ and let $b_i(v_i) = (b_i^1(v_i^1), \dots, b_i^k(v_i^k))$. For all $i \in \{1, \dots, n\}$ and for all $l \in \{1, \dots, k\}$, let us sort all values $b_j^l(v_j), j \neq i$ in a decreasing order and let $c_{-i} = (c_{-i}^1, \dots, c_{-i}^k)$ denote the first k values of these sorted values. For all $i \in \{1, \dots, n\}$ and for all $l \in \{1, \dots, k\}$, let the functions V_0 and V_i be defined as follows.*

$$\begin{array}{llll} V_0 : & (\mathbb{R}^k)^n & \to & (\mathbb{R}^k)^n \\ & v & \mapsto & q^V(v) \end{array}$$

$$\begin{aligned} &\textit{where } q^V(v) = \big(q^V(v)_1, \dots, q^V(v)_n\big), \\ &\forall l \in \{1, \dots, k\}, q_l^V(v) = \big(q_l^V(v)_1, \dots, q_l^V(v)_n\big) \\ &\textit{and } q_l^V(v)_i = \begin{cases} 1 & \textit{if} \quad b_i^l(v_i^l) > c_{-i}^{k-l+1} \\ 0 & \textit{if} \quad b_i^l(v_i^l) < c_{-i}^{k-l+1} \end{cases} \qquad \forall l \in \{1, \dots, k\} \end{aligned}$$

$$\begin{array}{llll} V_i : & (\mathbb{R}^k)^n & \to & \mathbb{R} \\ & v & \mapsto & \sum_{j=1}^{t_i} c_{-i}^{k-t_i+j} \end{array}$$

$$\textit{where } v = (v_1, \dots, v_n) \textit{ and } t_i = \sum_{l=1}^{k} q_l^V(v)_i$$

In a multi-unit vickrey auction, bidder i wins the l'th object if his bid $b_i^l(v_i^l)$ is higher than the l'th lowest value of the sorted c_{-i}. In case bidder i wins this l'th object he has to pay c_{-i}^{k-l+1} for it.

Observation 2.11. *There are some connections between the presented single-unit and multi-unit auctions. First consider the discriminatory auction in the case $k = 1$. In this case we see that the discriminatory auction is identical to the single-unit first price auction. The set C contains only one value, the highest bid received by all bidders. Thus, for bidder i the case $b_i(v_i) \in C$ is indentical to the case $b_i(v_i) > \max_{j \neq i} b_j(v_j)$. Also the pricing rule is the same, in both cases the winner of the auction pays his own bid. Let us now consider the other two presented multi-unit auctions, the uniform price and the vickrey auction. If we set $k = 1$ we see that both of these auctions are equivalent to the single-unit second price auction. Again, the winner of the auction in all three auctions is the bidder submitting the highest bid. We see that in the case $k = 1$, $\max_{j \neq i} b_j(v_j) < b_i(v_i)$, $b_i(v_i) \in C$ and $c_i^1 < b_i(v_i)$ are all three different notations for the same rule. It is easy to see that the same holds for the payment rule of all three auctions in the case $k = 1$.*

2.4 Revelation principle

The auction formats introduced in the last section have some common characteristics: Each time, an auction of the form $A = (A_0; A_1 \dots, A_n)$ is set up, with an allocation rule A_0 and a payment rule $(A_1, \ldots, A_n)$. But in addition we have bidding functions b, transforming the values v of the bidders into the bids. Of course the presence of the bidding functions makes the analysis of auction formats more complicated. Auction formats without a bidding function or with a bidding function as defined below would make the analysis easier.

Definition 2.12 (Revealing bidding function)**.** *Let b_R be a function of type $\mathbb{R} \to \mathbb{R}$ defined as follows.*

$$b_R : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ v & \mapsto & v \end{array}$$

Thus, by b_R we characterize the identity function.

An auction format in which the revealing bidding function is used is equal to an auction format without bidding function.

Definition 2.13 (Direct Auctions)**.** *We will call all auction formats in which only revealing bidding functions are used, direct auctions.*

The following lemma will help us for our further analysis, since it will allow us to restrict ourselves to direct auctions.

Lemma 2.14 (Revelation Principle)**.** *For any auction format $A = (A_0; A_1, \ldots, A_n)$ with corresponding bidding function b_A, there exists an "outcome equivalent" direct auction $B = (B_0; B_1, \ldots, B_n)$ with revealing bidding function b_R in the following sense: The outcome in both auctions is the same, i.e. all participants have the same probability of winning and the same payment obligations in both auctions.*

Proof. This relevation principle is based on Myerson[30]. He proves this result "in the more general context of Bayesian collective choice problems"[31]. In order to see why the revelation principle holds in our setting we just construct the functions $B_0(\cdot)$ and $B_1(\cdot), \ldots, B_n(\cdot)$ as compositions of $A_0(b(\cdot))$ and $A_1(b(\cdot)), \ldots, A_n(b(\cdot))$ respectively such that

$$B_0(\cdot) = A_0(b(\cdot))$$
$$\forall i \in \{1, \ldots, n\}, B_i(\cdot) = A_i(b(\cdot))$$

[30] Myerson, R.: Incentive Compatibility and the Bargaining Problem, Econometrica, Volume 47, Issue 1 (Jan., 1979), pp. 61-74.

[31] Myerson, R.: Optimal Auction Design, Mathematics of Operations Research, Vol. 6, No. 1 (Feb., 1981), p. 62.

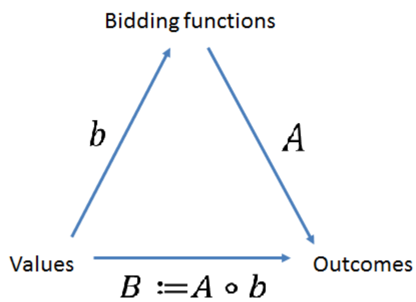

Figure 2.1: Revelation Principle[32]

□

Thus, for the analysis of the auction formats we will restrict ourselves to direct auctions.

2.5 Reserve price

In the previous sections and in the presented examples of single-unit and multi-unit auctions, the seller played a passive role. We now want to characterise a seller with the possibility to set a reserve price in the auction. A reserve price can be seen as a bid by the seller. For example, in a first price auction (see Definition 2.4), a bidder will win the auction if his bid is higher than all other submitted bids *and* higher than the reserve price set by the seller. Thus, there is a possibility that the object will not be auctioned-off (if all bids are below the reserve price). Let us model the first price auction with a reserve price.

Definition 2.15. *Let n be a natural number. For all i between 1 and n, let v_i be a real number and let $r \in \mathbb{R}$. For all $i \in \{1, \dots, n\}$, let the function b_i be of type $\mathbb{R} \to \mathbb{R}$ and let the functions F_0 and $F_1, \dots, F_n$ be defined as follows.*

$$\begin{array}{llll} F_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & v & \mapsto & q^F(v) \end{array}$$

$$\textit{where } q^F(v) = \left(q^F(v)_1, \dots, q^F(v)_n\right)$$
$$\textit{and } \forall i \in \{1, \dots, n\}, q^F(v)_i = \begin{cases} 1 & \textit{if} \quad b_i(v_i) > \max\left(r, \max_{j \neq i} b_j(v_j)\right) \\ 0 & \textit{if} \quad b_i(v_i) < \max\left(r, \max_{j \neq i} b_j(v_j)\right) \end{cases}$$

$$\begin{array}{llll} F_i: & \mathbb{R}^n & \to & \mathbb{R} \\ & v & \mapsto & \begin{cases} b_i(v_i) & \textit{if} \quad b_i(v_i) > \max\left(r, \max_{j \neq i} b_j(v_j)\right) \\ 0 & \textit{if} \quad b_i(v_i) < \max\left(r, \max_{j \neq i} b_j(v_j)\right) \end{cases} \end{array}$$

$$\textit{where } v = (v_1, \dots, v_n)$$

[32] Cf. Krishna, V.: Auction Theory, Academic Press, 2009, p. 63.

3 Revenue equivalence theorem

3.1 Preliminaries

The revenue equivalence theorem *(RET)* is one of the main classical results in auction theory. In this section we revisit this result. After introducing a technical lemma we will first discuss the original version of the revenue equivalence theorem by Vickrey. Then, we will also characterize two generalized versions of the theorem by Riley/Samuelson and Myerson. Finally, we will present the revenue equivalence theorem in an even more general framework.

Lemma 3.1. *Let f be a function of type $\mathbb{R}^2 \to \mathbb{R}$ and let g be a function of type $\mathbb{R} \to \mathbb{R}$. Assume that f is differentiable. Then assertion 2 follows from assertion 1.*

1. $\forall x, y \in \mathbb{R}, x \neq y \Rightarrow f(x, y) - g(y) < f(x, x) - g(x)$.
2. $\forall x, y \in \mathbb{R}, \int_x^y \frac{\partial f}{\partial x_1}(z, z)dz = f(y, y) - g(y) - f(x, x) + g(x)$.

Proof. Let us define the function k as follows.

$$\begin{array}{rccl} k: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & f(x, x) - g(x) \end{array}$$

Let x and y be distinct real numbers. By assumption 1 we have

$$\big(f(y, x) - f(x, x) + f(x, x)\big) - g(x) < f(y, y) - g(y)$$

so we obtain.

$$f(y, x) - f(x, x) < k(y) - k(x) \tag{1}$$

Since inequality (1) holds for all distinct $x, y \in \mathbb{R}$, we can swap x and y and obtain.

$$f(x, y) - f(y, y) < k(x) - k(y)$$

We multiply the inequality above by -1 and obtain.

$$k(y) - k(x) < f(y, y) - f(x, y) \tag{2}$$

By combining the two inequalities (1) and (2) we conclude.

$$f(y, x) - f(x, x) < k(y) - k(x) < f(y, y) - f(x, y)$$

Let us now assume that $x < y$ and let us divide the double inequality above by $y - x$.

$$\frac{f(y - x) - f(x, x)}{y - x} < \frac{k(y) - k(x)}{y - x} < \frac{f(y, y) - f(x, y)}{y - x}$$

Since f is differentiable, one can prove by this double inequality that the function k is differentiable and that

$$\forall z \in \mathbb{R}, k'(z) = \frac{\partial f}{\partial x_1}(z, z).$$

So the following holds for all real numbers x and y.

$$\int_x^y \frac{\partial f}{\partial x_1}(z,z)dz = f(y,y) - g(y) - f(x,x) + g(x)$$

□

3.2 Revenue equivalence theorem by Vickrey

Vickrey was not only the one who established the second price auction as a new sealed bid auction format, he was also the first who discovered the revenue equivalence of different auction formats in his now classical paper of 1961. Basically, Vickrey showed that under some assumptions regarding the distribution of the bidders, the progressive auction and the dutch auction are equivalent "in terms of average expected outcomes."[33]. Vickrey's assumption is that the values by the bidders are independent and indentically drawn from an uniform distribution ranging from 0 to 1.

3.3 Revenue equivalence theorem by Riley and Samuelson

Riley and Samuelson carry on Vickrey's assumption regarding the distribution of the valuation of the bidders. In the beginning of their paper, they state their "IID assumption" which they use for their further theorems.

Definition 3.2 (IID assumption by Riley and Samuelson)**.** *The reservation values of the parties are independent and identically distributed, drawn from the common distribution $F(v)$ with $F(\underline{v}) = 0$, $F(\overline{v}) = 1$ and $F(v)$ strictly increasing and differentiable over the interval $[\underline{v}, \overline{v}]$.*[34]

While Vickrey showed the revenue equivalence only for the progressive and dutch auctions, Riley and Samuelson define a certain family of auction formats for which the following four assumptions are satisfied.

1. Each bidder can make any bid above some reserve price set by the seller.
2. The bidder with the highest bid wins the object.
3. Each bidder is treated in the same way, they are anonymous.
4. There exists a common equilibrium bidding stragegy. Each bidder i submits a bid b_i, which is a strictly increasing function of his value v_i.

We will only present the theorem of Riley and Samuelson characterizing the assumptions they made in our new formalism. The proof itself is then just a special case of part a) of the generalized version in Section 3.5.

[33] Vickrey, W.: Counterspeculation, Auctions and Competitive Sealed Tenders, Journal of Finance, 1961, p. 30.

[34] See: Riley, J., Samuelson, W.: Optimal Auctions, The American Economic Review, 1981, p. 381.

Theorem 3.3 (RET by Riley and Samuelson)**.** *Let $\underline{v}, \overline{v} \in \mathbb{R}$ and $\underline{v} \leq \overline{v}$. Let n be a natural number and $V = (V_1, \ldots, V_n)$ be a vector consisting of n real-valued random variables taking values in $[\underline{v}, \overline{v}]$. Let A_0 and B_0 be functions of type $([\underline{v}, \overline{v}])^n \to \mathbb{R}^n$ defined as follows:*

$$\begin{array}{rlll} X_0 : & ([\underline{v}, \overline{v}])^n & \to & \mathbb{R}^n \\ & x & \mapsto & q^X(x) \end{array}$$

where X represents A or B, $q^X(x) = \big(q^X(x)_1, \ldots, q^X(x)_n\big)$, $\forall i \in \{1, \ldots, n\}, 0 \leq q^X(x)_i$ and $\sum_{i=1}^n q^X(x)_i \leq 1$.

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be functions of type $\mathbb{R}^n \to \mathbb{R}$. For all $i \in \{1, \ldots, n\}$, let the function u_i be defined as follows.

$$\begin{array}{rlll} u_i : & [\underline{v}, \overline{v}] & \to & [\underline{v}, \overline{v}] \\ & v & \mapsto & v \end{array}$$

For X representing A or B and for every natural number i between 1 and n let us assume that the functions e_i^X and a_i^X below are well-defined.

$$\begin{array}{rlll} e_i^X : & [\underline{v}, \overline{v}] & \to & \mathbb{R} \\ & v & \mapsto & E(X_i(V[i \to v])) \end{array}$$

$$\begin{array}{rlll} a_i^X : & ([\underline{v}, \overline{v}])^2 & \to & \mathbb{R} \\ & x, y & \mapsto & E(u_i(x) \cdot q^X(V[i \to y]_i)) \end{array}$$

where $V[i \to v] := (V_1, \ldots, V_{i-1}, v, V_{i+1}, \ldots, V_n)$

We assume the following.

1. *For all i between 1 and n, the function a_i^X is differentiable.*
2. *The random variables $V_1, \ldots, V_n$ are mutually independent.*
3. *For all i between 1 and n, let f be a density function for V_i, where $\int_{\underline{v}}^{\overline{v}} f(x)dx = 1$.*
4. *For all $i \in \{1, \ldots, n\}$ we have $a_i^A = a_i^B$.*
5. *Let $(v_i)_{i \in \{1, \ldots, n\}}$ be a family of real numbers, where $\forall i \in \{1, \ldots, n\}, v_i \in [\underline{v}, \overline{v}]$, such that $\sum_{i=1}^n e_i^A(v_i) = \sum_{i=1}^n e_i^B(v_i)$.*
6. *For all i between 1 and n and for all distinct real numbers $x, y \in [\underline{v}, \overline{v}]$,*

$$a_i^X(x, y) - e_i^X(y) < a_i^X(x, x) - e_i^X(x)$$

The following can be deduced from the assumptions above.

$$E(\sum_{i=1}^n A_i(V)) = E(\sum_{i=1}^n B_i(V))$$

As explained before, Riley and Samuelson use the IID assumption for their analysis. In this assumption, they use the interval $[\underline{v}, \overline{v}]$ as a bound for the values v_i of all bidders $i \in \{1, \ldots, n\}$. Furthermore, they only take the values v_i into account for the valuation of the bidders. Thus, the function u_i we defined in the theorem is just the identity function.

3.4 Revenue equivalence theorem by Myerson

Riley and Samuelson worked on their paper in the same time as Myerson wrote his paper on "Optimal Auction Design". Myerson uses a more technical approach in his paper than Riley and Samuelson. As explained before in Section 2.4, Myerson discovered the Relevation Principle (see Lemma 2.1). By this theorem he was able to analyse auction formats and theorems without using bidding functions. Furthermore, in contrary to Vickrey and Riley/Samuelson, Myerson does not use the IID assumption. In his concept, from the point of view of the seller, the value of each bidder is a random variable with an individual distribution / density function. But also Myerson uses bounds for the values of the bidders, even though he uses a more general approach: Just like he introduced individual density functions, he allows for individual intervals $[\alpha_i, \beta_i]$ as bounds for the values of the bidders. Finally, Myerson introduces more complex functions u_i to characterize the bidder's valuation. While in Riley and Samuelson's paper the function u_i of bidder i is only the identity function, Myerson takes also the values of the other bidders into account, in form of so called "revision effect" functions.

Definition 3.4 (Revision effect functions). *Let n be a natural number. For all i between 1 and n, let the function τ_i be of type $[\alpha_i, \beta_i] \to \mathbb{R}$.*

Again, we will present now only the theorem with the definitions of the functions and the assumptions. The proof of the result can be easily derived from the proof of the generalized version below.

Theorem 3.5 (RET by Myerson). *Let n be a natural number and for all i between 1 and n, let $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i \leq \beta_i$. Let $V = (V_1, \ldots, V_n)$ be a vector consisting of n real-valued random variables. Let A_0 and B_0 be functions of type $\mathbb{R}^n \to \mathbb{R}^n$ defined as follows:*

$$
\begin{array}{llll}
X_0 : & [\alpha, \beta] & \to & \mathbb{R}^n \\
 & x & \mapsto & q^X(x)
\end{array}
$$

where X represents A or B, $q^X(x) = \big(q^X(x)_1, \ldots, q^X(x)_n\big)$, $\forall i \in \{1, \ldots, n\}, 0 \leq q^X(x)_i$ and $\sum_{i=1}^n q^X(x)_i \leq 1$, and where $[\alpha, \beta] := [\alpha_1, \beta_1] \times, \ldots, \times [\alpha_n, \beta_n]$.

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be functions of type $\mathbb{R}^n \to \mathbb{R}$. For all $i \in \{1, \ldots, n\}$, *let the function u_i be defined as follows.*

$$\begin{array}{llll} u_i: & [\alpha, \beta] & \to & \mathbb{R} \\ & v & \mapsto & v_i + \sum_{j \in \{1,\ldots,n\}, j \neq i} \tau_j(v_j) \\ & & & \text{where } v = (v_1, \ldots, v_n). \end{array}$$

For X representing A or B and for every natural number i between 1 and n let us assume that the functions e_i^X and a_i^X below are well-defined.

$$\begin{array}{llll} e_i^X: & [\alpha_i, \beta_i] & \to & \mathbb{R} \\ & v & \mapsto & E(X_i(V[i \to v])) \end{array}$$

$$\begin{array}{llll} a_i^X: & ([\alpha_i, \beta_i])^2 & \to & \mathbb{R} \\ & x, y & \mapsto & E(u_i(V[i \to x]) \cdot q^X(V[i \to y]_i)) \\ & & & \\ & & & \text{where } V[i \to v] := (V_1, \ldots, V_{i-1}, v, V_{i+1}, \ldots, V_n) \end{array}$$

We assume the following.

1. *For all i between 1 and n, the function a_i^X is differentiable.*
2. *The random variables $V_1, \ldots, V_n$ are mutually independent.*
3. *For all i between 1 and n, let f_i be a density function for V_i, where $\int_{\alpha_i}^{\beta_i} f_i(x)dx = 1$.*
4. *For all $i \in \{1, \ldots, n\}$ we have $a_i^A = a_i^B$.*
5. *Let $(v_i)_{i \in \{1,\ldots,n\}}$ be a family of real numbers, where $\forall i \in \{1, \ldots, n\}, v_i \in [\alpha_i, \beta_i]$, such that $\sum_{i=1}^n e_i^A(v_i) = \sum_{i=1}^n e_i^B(v_i)$.*
6. *For all i between 1 and n and for all distinct real numbers x_i and $y_i \in [\alpha_i, \beta_i]$,*

$$a_i^X(x_i, y_i) - e_i^X(y_i) < a_i^X(x_i, x_i) - e_i^X(x_i)$$

The following can be deduced from the assumptions above.

$$E(\sum_{i=1}^n A_i(V)) = E(\sum_{i=1}^n B_i(V))$$

As mentioned before, we use a new formalism and way of proof for the theorem. Myerson, for example needs in fact the bounds $[\alpha, \beta]$ for his proof, since he shows that the payoff for the seller is determined only by the function X_0 and the payoff for all bidders at the lower bound α. We will now show, among other improvements, that we do not need to use bounds for the values of the bidders in order to proof the revenue equivalence theorem.

3.5 Generalization of the revenue equivalence theorem

After revisiting the different versions of the revenue equivalence theorem we now want to generalize the existing results. Let us first prove the result and then interpret the improvements of this theorem compared to the earlier versions.

Theorem 3.6. *Let n be a natural number and $V = (V_1, \dots, V_n)$ be a vector consisting of n real-valued random variables. Let A_0 and B_0 be functions of type $\mathbb{R}^n \to \mathbb{R}^n$ defined as follows:*

$$\begin{array}{llll} X_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & q^X(x) \end{array}$$

where X represents A or B, $q^X(x) = \big(q^X(x)_1, \dots, q^X(x)_n\big)$, $\forall i \in \{1, \dots, n\}, 0 \le q^X(x)_i$ and $\sum_{i=1}^n q^X(x)_i \le 1$.

Let $A_1, \dots, A_n$, $B_1, \dots, B_n$ and $u_1, \dots, u_n$ be functions of type $\mathbb{R}^n \to \mathbb{R}$. For X representing A or B and for every natural number i between 1 and n let us assume that the functions e_i^X, p_i^X and a_i^X below are well-defined.

$$\begin{array}{llll} e_i^X: & \mathbb{R} & \to & \mathbb{R} \\ & v & \mapsto & E(X_i(V[i \to v])) \end{array}$$

$$\begin{array}{llll} p_i^X: & \mathbb{R} & \to & [0,1] \\ & v & \mapsto & E(q^X(V[i \to v])_i) \end{array}$$

$$\begin{array}{llll} a_i^X: & \mathbb{R}^2 & \to & \mathbb{R} \\ & x, y & \mapsto & E(u_i(V[i \to x]) \cdot q^X(V[i \to y])_i) \end{array}$$

where $V[i \to v] := (V_1, \dots, V_{i-1}, v, V_{i+1}, \dots, V_n)$

We assume the following.

1. *For all i between 1 and n, the function a_i^X is differentiable.*
2. *The random variables $V_1, \dots, V_n$ are mutually independent.*
3. *For all i between 1 and n, let f_i be a density function for V_i. Let $J_1, \dots, J_k$ be a partition of $\{1, \dots, n\}$ complying with the following.*
   $\forall i, j \in \{1, \dots, n\}, f_i = f_j \Rightarrow \exists l \in \{1, \dots, k\}, i, j \in J_l$. Also, for all l between 1 and k, let the function f^l be defined as follows.
   $\forall i \in \{1, \dots, n\}, i \in J_l \Rightarrow f^l := f_i$.
4. *For all $l \in \{1, \dots, k\}$ we have $\sum_{i \in J_l} p_i^A = \sum_{i \in J_l} p_i^B$,*
   $\sum_{i \in J_l} a_i^A = \sum_{i \in J_l} a_i^B$ and $\sum_{i \in J_l} \frac{\partial a_i^A}{\partial x_1} = \sum_{i \in J_l} \frac{\partial a_i^B}{\partial x_1}$.
5. *Let $(v^l)_{l \in \{1, \dots, k\}}$ be a family of real numbers such that*
   $\sum_{l=1}^k \sum_{i \in J_l} e_i^A(v^l) = \sum_{l=1}^k \sum_{i \in J_l} e_i^B(v^l)$.

6. *For all i between 1 and n and for all distinct real numbers x and y,*

$$a_i^X(x, y) - e_i^X(y) < a_i^X(x, x) - e_i^X(x)$$

The following can be deduced from the assumptions above.

a) $E(\sum_{i=1}^n A_i(V)) = E(\sum_{i=1}^n B_i(V))$

b) $\sum_{i=1}^n E(q^A(V)_i) = \sum_{i=1}^n E(q^B(V)_i)$

c) $\sum_{i=1}^n e_i^A = \sum_{i=1}^n e_i^B$

d) *If there exists $v \in \mathbb{R}$ and $l \in \{1, \ldots, k\}$ such that $\sum_{i \in J_l} e_i^A(v) = \sum_{i \in J_l} e_i^B(v)$, then $\sum_{i \in J_l} e_i^A = \sum_{i \in J_l} e_i^B$.*

Proof. First notice that we can apply Lemma 3.1 for all i between 1 and n for the functions a_i^X and e_i^X, when we replace the functions f and g with a_i^X and e_i^X respectively, since by assumption 1, a_i^X is is differentiable and since the types of a_i^X and e_i^X do not restrict the application of the lemma.

a) Since assumption 6 is equivalent to assertion 1 of Lemma 3.1, we can use assertion 2 of the lemma. Then the following holds for all $i \in \{1, \ldots, n\}$ and for all real numbers x and y.

$$e_i^X(x) = \int_x^y \frac{\partial a_i^X}{\partial x_1}(z, z)dz - a_i^X(y, y) + e_i^X(y) + a_i^X(x, x) \tag{3}$$

By basic probability theory $E(\sum_{i=1}^n X_i(V)) = \sum_{i=1}^n E(X_i(V))$ (see Appendix, Definition A.18). For all $i \in \{1, \ldots, n\}$, we use the density function f_i of V_i, defined in assumption 3. So $E(X_i(V)) = E(X_i(V[i \to V_i])) = \int_{\mathbb{R}} e_i^X(x)f_i(x)dx$ (see Appendix, Definition A.14) by independence in assumption 2. Therefore, using equation (3), for all i between 1 and n, the following holds for all real numbers y.

$$E(\sum_{i=1}^n X_i(V)) = \sum_{i=1}^n \int_{\mathbb{R}} \left(e_i^X(y) - a_i^X(y, y) + a_i^X(x, x) + \int_x^y \frac{\partial a_i^X}{\partial x_1}(z, z)dz \right) f_i(x)dx$$

We now use the partition of the set $\{1, \ldots, n\}$ (see assumption 3). Then the equation above yields for all real numbers y.

$$\begin{aligned} E(\sum_{i=1}^n A_i(V)) &= \sum_{i=1}^n \int_{\mathbb{R}} \left(e_i^A(y) - a_i^A(y, y) + a_i^A(x, x) + \int_x^y \frac{\partial a_i^A}{\partial x_1}(z, z)dz \right) f_i(x)dx \\ &= \sum_{l=1}^k \sum_{i \in J_l} \int_{\mathbb{R}} \left(e_i^A(y) - a_i^A(y, y) + a_i^A(x, x) + \int_x^y \frac{\partial a_i^A}{\partial x_1}(z, z)dz \right) f_i(x)dx \end{aligned}$$

We use linearity of integration and summation and the fact that $\forall i \in \{1, \ldots, n\}, \int_{\mathbb{R}} f_i(x)dx = 1$ (see Appendix, Definition A.8). Since the equation above holds for all y, we use the family $(v^l)_{l\in\{1,\dots,k\}}$ defined in assumption 5 and obtain by using also assumptions 3 and 4.

$$\begin{aligned}
E(\sum_{i=1}^{n} A_i(V)) &= \sum_{l=1}^{k}\sum_{i\in J_l}\int_{\mathbb{R}}\left(e_i^A(v^l) - a_i^A(v^l,v^l) + a_i^A(x,x) + \int_x^{v^l}\frac{\partial a_i^A}{\partial x_1}(z,z)dz\right)f^l(x)dx \\
&= \sum_{l=1}^{k}\sum_{i\in J_l} e_i^A(v^l) - \sum_{l=1}^{k}\sum_{i\in J_l} a_i^A(v^l,v^l) + \sum_{l=1}^{k}\sum_{i\in J_l}\int_{\mathbb{R}}\left(a_i^A(x,x) + \int_x^{v^l}\frac{\partial a_i^A}{\partial x_1}(z,z)dz\right)f^l(x)dx \\
&= \sum_{l=1}^{k}\sum_{i\in J_l} e_i^A(v^l) - \sum_{l=1}^{k}\sum_{i\in J_l} a_i^A(v^l,v^l) + \sum_{l=1}^{k}\int_{\mathbb{R}}\left(\sum_{i\in J_l} a_i^A(x,x) + \int_x^{v^l}\sum_{i\in J_l}\frac{\partial a_i^A}{\partial x_1}(z,z)dz\right)f^l(x)dx \\
&= \sum_{l=1}^{k}\sum_{i\in J_l} e_i^B(v^l) - \sum_{l=1}^{k}\sum_{i\in J_l} a_i^B(v^l,v^l) + \sum_{l=1}^{k}\int_{\mathbb{R}}\left(\sum_{i\in J_l} a_i^B(x,x) + \int_x^{v^l}\sum_{i\in J_l}\frac{\partial a_i^B}{\partial x_1}(z,z)dz\right)f^l(x)dx \\
&= E(\sum_{i=1}^{n} B_i(V))
\end{aligned}$$

b) By assumption 4 and by using the partition of the set $\{1,\dots,n\}$ we obtain the following.

$$\begin{aligned}
\sum_{i=1}^{n} E(q^A(V)_i) &= \sum_{i=1}^{n} E(q^A(V[i\to V_i])_i) = \sum_{i=1}^{n}\int_{\mathbb{R}} p_i^A(x)f_i(x)dx \\
&= \sum_{l=1}^{k}\sum_{i\in J_l}\int_{\mathbb{R}} p_i^A(x)f^l(x)dx = \sum_{l=1}^{k}\int_{\mathbb{R}}\sum_{i\in J_l} p_i^A(x)f^l(x)dx \\
&= \sum_{l=1}^{k}\int_{\mathbb{R}}\sum_{i\in J_l} p_i^B(x)f^l(x)dx = \sum_{i=1}^{n} E(q^B(V)_i)
\end{aligned}$$

c) We use the family $(v^l)_{l\in\{1,\dots,k\}}$ defined in assumption 5 and equation (3). Then we obtain the following for all $x\in\mathbb{R}$.

$$\sum_{l=1}^{k}\sum_{i\in J_l} e_i^X(x) = \sum_{l=1}^{k}\int_x^{v^l}\sum_{i\in J_l}\frac{\partial a_i^X}{\partial x_1}(z,z)dz - \sum_{l=1}^{k}\sum_{i\in J_l} a_i^X(v^l,v^l) + \sum_{l=1}^{k}\sum_{i\in J_l} e_i^X(v^l) + \sum_{l=1}^{k}\sum_{i\in J_l} a_i^X(x,x)$$

Thus we conclude, using also assumption 4.

$$\begin{aligned}
\sum_{l=1}^{k}\sum_{i\in J_l} e_i^A(x) &= \sum_{l=1}^{k}\int_x^{v^l}\sum_{i\in J_l}\frac{\partial a_i^A}{\partial x_1}(z,z)dz - \sum_{l=1}^{k}\sum_{i\in J_l} a_i^A(v^l,v^l) + \sum_{l=1}^{k}\sum_{i\in J_l} e_i^A(v^l) + \sum_{l=1}^{k}\sum_{i\in J_l} a_i^A(x,x) \\
&= \sum_{l=1}^{k}\int_x^{v^l}\sum_{i\in J_l}\frac{\partial a_i^B}{\partial x_1}(z,z)dz - \sum_{l=1}^{k}\sum_{i\in J_l} a_i^B(v^l,v^l) + \sum_{l=1}^{k}\sum_{i\in J_l} e_i^B(v^l) + \sum_{l=1}^{k}\sum_{i\in J_l} a_i^B(x,x) \\
&= \sum_{l=1}^{k}\sum_{i\in J_l} e_i^B(x)
\end{aligned}$$

Since this is true for all real numbers x we obtain

$$\sum_{i=1}^{n} e_i^A = \sum_{l=1}^{k} \sum_{i \in J_l} e_i^A = \sum_{l=1}^{k} \sum_{i \in J_l} e_i^B = \sum_{i=1}^{n} e_i^B.$$

d) Let $l \in \{1, \dots, k\}$ and $v \in \mathbb{R}$ such that

$$\sum_{i \in J_l} e_i^A(v) = \sum_{i \in J_l} e_i^B(v). \tag{4}$$

Analogous to the proof of *c*), we use equation (3) and obtain for all $x \in \mathbb{R}$.

$$\sum_{i \in J_l} e_i^X(x) = \int_x^v \sum_{i \in J_l} \frac{\partial a_i^X}{\partial x_1}(z,z)dz - \sum_{i \in J_l} a_i^X(v,v) + \sum_{i \in J_l} e_i^X(v) + \sum_{i \in J_l} a_i^X(x,x)$$

Now by using assumption 4 and equation (4), we conclude.

$$\begin{aligned}
\sum_{i \in J_l} e_i^A(x) &= \int_x^v \sum_{i \in J_l} \frac{\partial a_i^A}{\partial x_1}(z,z)dz - \sum_{i \in J_l} a_i^A(v,v) + \sum_{i \in J_l} e_i^A(v) + \sum_{i \in J_l} a_i^A(x,x) \\
&= \int_x^v \sum_{i \in J_l} \frac{\partial a_i^B}{\partial x_1}(z,z)dz - \sum_{i \in J_l} a_i^B(v,v) + \sum_{i \in J_l} e_i^B(v) + \sum_{i \in J_l} a_i^B(x,x) \\
&= \sum_{i \in J_l} e_i^B(x)
\end{aligned}$$

Since this is true for all real numbers x we obtain

$$\sum_{i \in J_l} e_i^A = \sum_{i \in J_l} e_i^B.$$

□

This is the generalized version of the revenue equivalence theorem. The first generalization we can see directly is that we do not use bounds anymore. All values of the bidders $i \in \{1, \dots, n\}$ are arbitrary real numbers. Also, we do not make any assumptions or restrictions regarding the functions u_i. So we allow Riley/Samuelson's version of the functions u_i (just the identity functions) and Myerson's version with the revision effect functions. Furthermore, we use a partition of the set $\{1, \dots, n\}$ in order to define the density functions of the random variables V_i. By this partition we allow that there may be certain groups of bidders with the same density function, all bidders may have the same density function (like in Riley/Samuelson), or each bidder has a different density function (like in Myerson's version).

In addition, in assumption 4 we do not require anymore that for all i between 1 and n, $a_i^A = a_i^B$. In the proof we can see that it is sufficient that for all l between 1 and k, $\sum_{i \in J_l} p_i^A = \sum_{i \in J_l} p_i^B$ (for the proof of part *b*),

$\sum_{i\in J_l} a_i^X = \sum_{i\in J_l} a_i^X$ and $\sum_{i\in J_l} \frac{\partial a_i^A}{\partial x_1} = \sum_{i\in J_l} \frac{\partial a_i^B}{\partial x_1}$, which is a far weaker condition than before.

All three version of the theorem have assumption 6 in common. Myerson calls this assertion the *incentive-compatibility* condition for the bidders. By $a(x,x) - e(x)$ we define the expected payoff of the bidders. Here, $a(x,x)$ defines the expected profit, i.e. the expected valuation for the object subject to the probability of winning it. The function $e(x)$ represents the expected payments in the auction, thus the difference of these two functions results in the expected payoffs by the bidders. By assumption 6 we ensure that a bidder with true value $x \in \mathbb{R}$ always has a greater expected payoff when he submits this real value x to the seller and not any other value $y \in \mathbb{R}$. So there is no incentive to lie about one's personal true valuation for the object.

Finally we prove three further assertions in addition to the fact that the expected payoff of the seller in the two auctions A and B is the same. In b) we show that the sums of the expected probability to win is the same in the two auctions from the point of view of the seller. In c) we see that also the sums of the expected payment functions e_i^X are the same for the two auctions. If there exists in addition a real number v and a subgroup J_l of the partition such that the sums of the expected payment functions coincide in that point v, then we show in d) that the sums of the functions coincide everywhere, $\sum_{i\in J_l} e_i^A = \sum_{i\in J_l} e_i^B$.

We now want go back to the technical lemma from the beginning of the section. As mentioned above, we characterize the expected payoff of the bidders by $a(x,x) - e(x)$. If we assume now that the functions $u(x)$ are the identity function (like in the version of Riley and Samuelson), then we can describe the expected payoff by $x \cdot a(x) - e(x)$. In this case, we can simplify the technical lemma which also allows us to derive one further result.

Lemma 3.7. *Let f and g be two functions of type $\mathbb{R} \to \mathbb{R}$ and let us assume that the function f is continuous. Then assertions 1 and (2a$\wedge$2b) are equivalent.*

1. $\forall x, y \in \mathbb{R}, x \neq y \Rightarrow xf(y) - g(y) < xf(x) - g(x)$.
2. (a) *f is strictly increasing, i.e.* $\forall x, y \in \mathbb{R}, x < y \Rightarrow f(x) < f(y)$.
   (b) $\forall x, y \in \mathbb{R}, \int_x^y f = yf(y) - g(y) - xf(x) + g(x)$.

Proof. "1 $\Rightarrow$ (2a$\wedge$2b)"
Let us define the function k as follows.

$$k: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & xf(x) - g(x) \end{array}$$

Let x and y be distinct real numbers. By assumption 1 we have

$$(y - x + x)f(x) - g(x) < yf(y) - g(y)$$

so we obtain.

$$(y - x)f(x) < k(y) - k(x)$$

Therefore, $(x-y)f(y) < k(x)-k(y)$ holds too by swapping x and y in the inequality above. We multiply the second inequality by -1 and obtain.

$$k(y)-k(x) < (y-x)f(y)$$

Combining the two inequalities above yields the following double inequality.

$$(y-x)f(x) < k(y)-k(x) < (y-x)f(y)$$

Let us now assume that $x < y$. Let us divide the double inequality by $y-x$, this yields the double inequality below.

$$f(x) < \frac{k(y)-k(x)}{y-x} < f(y) \tag{5}$$

This double inequality shows that the function f is strictly increasing. Since f is continuous, letting y approach x from above in the double inequality (5) shows that k is differentiable and that its derivative k' equals f. So the following holds for all real numbers x and y.

$$\int_x^y f = yf(y) - g(y) - xf(x) + g(x)$$

"(2a∧2b)⇒ 1"
Let $x, y \in \mathbb{R}$ and $x \neq y$. Since the function f is strictly increasing by assumption 2a we obtain.

$$\forall x, y \in \mathbb{R}, x < y \Rightarrow \int_x^y f(x)dt < \int_x^y f(t)dt \tag{6}$$

In the same way we obtain the following.

$$\forall x, y \in \mathbb{R}, y < x \Rightarrow \int_y^x f(t)dt < \int_y^x f(x)dt$$

Now we multiply the inequality above by -1 and conclude.

$$\int_x^y f(x)dt = -\int_y^x f(x)dt < -\int_y^x f(t)dt = \int_x^y f(t)dt \tag{7}$$

We combine inequalities 6 and 7 and obtain.

$$\forall x, y \in \mathbb{R}, x \neq y \Rightarrow \int_x^y f(x)dt < \int_x^y f(t)dt$$

Finally we conclude by using assumption 2b.

$$\begin{aligned}
yf(y) - g(y) &> xf(x) - g(x) + \int_x^y f(x)dt \\
&> xf(x) - g(x) + f(x)\int_x^y 1dt \\
&> xf(x) - g(x) + f(x)(y-x) \\
&> xf(x) - g(x) + yf(x) - xf(x) \\
&> yf(x) - g(x)
\end{aligned}$$

This proves assertion 1 of the lemma for all distinct real numbers x and y. □

Applying now this lemma in the generalized version of the revenue equivalence theorem in the described case where $u(x) = x$ we obtain that for all i between 1 and n, the functions a_i^X are strictly increasing. In this special case, the functions a_i^X only represent the probability to win from the point of view of bidder i. Furthermore, this lemma shows an equivalence between the *incentive compatibility* and the two assertions that the functions a_i^X are strictly increasing and the intermediate equation (3) we use in the proof. In addition, we do not need to assume anymore that the function f is differentiable. As we can see it in the proof it is sufficient to assume that the function f is continuous.

3.6 Comparison of auction formats

After deriving the generalized version of the revenue equivalence theorem we want to show its applicability by an example. We will consider the two standard auction formats; first price and second price auction, defined in Section 2.

Let n be a natural number and let us assume that for all bidders i between 1 and n, the value for the object lies between 0 and 100. Let us first compare the functions F_0 and S_0.

$$\begin{array}{rccl} F_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & v & \mapsto & q^F(v) \end{array}$$

$$\text{where } q^F(v) = \big(q^F(v)_1, \ldots, q^F(v)_n\big)$$
$$\text{and } \forall i \in \{1, \ldots, n\}, q^F(v)_i = \begin{cases} 1 & \text{if} \quad b_i(v_i) > \max_{j \neq i} b_j(v_j) \\ 0 & \text{if} \quad b_i(v_i) < \max_{j \neq i} b_j(v_j) \end{cases}$$

$$\begin{array}{rccl} S_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & v & \mapsto & q^S(v) \end{array}$$

$$\text{where } q^S(v) = \big(q^S(v)_1, \ldots, q^S(v)_n\big)$$
$$\text{and } \forall i \in \{1, \ldots, n\}, q^S(v)_i = \begin{cases} 1 & \text{if} \quad b_i(v_i) > \max_{j \neq i} b_j(v_j) \\ 0 & \text{if} \quad b_i(v_i) < \max_{j \neq i} b_j(v_j) \end{cases}$$

We see immediatley that the two functions are identical and thus we can obtain that for all i between 1 and n also the functions p_i^F and p_i^S are equivalent. Thereby, assumption 4 is valid.

We assumed that the value for all bidders lies in the interval $[0, 100]$ and in both auctions only the winner of the auction, i.e. the bidder with the highest bid, has to pay something. So we conclude that for all bidders the expected payments in both auctions equals zero when the value is zero.

$$\sum_{i=1}^{n} e_i^F(0) = 0 = \sum_{i=1}^{n} e_i^S(0)$$

Thus, also assumption 5 of the revenue equivalence theorem holds and we can apply the theorem for the two auction formats and conclude.

$$E(\sum_{i=1}^{n} F_i(V)) = E(\sum_{i=1}^{n} S_i(V))$$

That is, the expected payoff for the seller in the first price auction and in the second price auction is the same under the assumptions we made.

4 Multi-unit revenue equivalence theorem

4.1 Preliminaries

After discussing in detail the development of the revenue equivalence theorem and presenting the generalized version of it, we now want to investigate the multi-unit version of the RET. Again, we will first proof a technical lemma which will be then used for the theorem itself. We will also have a look at the earlier versions of the multi-unit revenue equivalence theorem. In fact, Vickrey was again the one who first characterized revenue equivalence for multi-unit auctions, but only by an example. He explaines that the expected payoff for the seller is the same whether he performs an auction where the winning bidders have to pay an amount equal to the lowest winning bid or an amount equal to the highest losing bid. The "real" multi-unit version of the revenue equivalence theorem is written by Krishna and Perry[35].

Lemma 4.1. *Let f be a function of type $(\mathbb{R}^k)^2 \to \mathbb{R}$ and let g be a function of type $\mathbb{R}^k \to \mathbb{R}$. Furthermore, assume that f is differentiable. Then assertion 2 follows from assertion 1.*

1. $\forall x, y \in \mathbb{R}^k, x \neq y \Rightarrow f(x,y) - g(y) < f(x,x) - g(x)$.
2. *Let* $a, b \in \mathbb{R}^k$. *Then,* $\forall s, t \in [0,1]$,

$$\int_s^t \frac{\partial f}{\partial x_1}(u^*, u^*)du = f(t^*, t^*) - g(t^*) - f(s^*, s^*) + g(s^*),$$

where $z^* = a + z(b-a)$.

Proof. Let $x, y \in \mathbb{R}^k$ be two distinct vectors. Let us define the function k as follows.

$$\begin{array}{llll} k: & [0,1] & \to & \mathbb{R} \\ & t & \mapsto & f(t^*, t^*) - g(t^*) \end{array}$$

Let $s, t \in [0,1]$ and $s \neq t$. By assertion 1 we obtain.

$$f(t^*, s^*) - g(s^*) < f(t^*, t^*) - g(t^*)$$

This inequality is equivalent to the following inequality below.

$$f(t^*, s^*) - f(s^*, s^*) + f(s^*, s^*) - g(s^*) < f(t^*, t^*) - g(t^*)$$

Thus, the following holds as well.

$$\forall s, t \in [0,1], s \neq t \Rightarrow f(t^*, s^*) - f(s^*, s^*) < k(t) - k(s) \qquad (8)$$

Since the inequality above holds for all $s, t \in [0,1]$, the following holds too by swapping s and t.

$$\forall s, t \in [0,1], s \neq t \Rightarrow f(s^*, t^*) - f(t^*, t^*) < k(s) - k(t)$$

[35] Krishna, V. and Perry, M.: Efficient Mechanism Design, 1998.

We multiply the inequality above by -1 and obtain.

$$k(t) - k(s) < f(t^*, t^*) - f(s^*, t^*) \tag{9}$$

We conclude by combining the two inequalities (8) and (9).

$$f(t^*, s^*) - f(s^*, s^*) < k(t) - k(s) < f(t^*, t^*) - f(s^*, t^*)$$

Let us assume that $s < t$ and let us divide the double inequality above by $t - s$.

$$\frac{f(t^*, s^*) - f(s^*, s^*)}{t - s} < \frac{k(t) - k(s)}{t - s} < \frac{f(t^*, t^*) - f(s^*, t^*)}{t - s}$$

Since f is differentiable, one can prove by this double inequality that the function k is differentiable and that

$$\forall u \in [0, 1], k'(u) = \frac{\partial f}{\partial x_1}(u^*, u^*).$$

Thus, the following holds for all $s, t \in [0, 1]$.

$$\int_s^t \frac{\partial f}{\partial x_1}(u^*, u^*) du = f(t^*, t^*) - g(t^*) - f(s^*, s^*) + g(s^*)$$

□

4.2 Multi-unit revenue equivalence theorem by Krishna and Perry

First we want to describe the model of multi-unit auctions which Krishna and Perry use for their theorem in their original derivation. There are $n \in \mathbb{N}$ bidders and $k \in \mathbb{N}$ identical objects to be auctioned-off. The value of each bidder i is represented by a $k-$vector $v_i = (v_i^1, \ldots, v_i^k)$, where v_i^k "represents the marginal value of obtaining the kth object and these are declining."[36] In the same way as in the previous section, we will present only the theorem and state the functions and assumptions in our new formalism. The proof is then just a special case of the proof of a) of the generalized version discussed below.

Theorem 4.2. *Let n and k be two natural numbers. For all i between 1 and n, let $V_i = (V_i^1, \ldots, V_i^k)$ be a random vector and let $V := (V_1, \ldots, V_n)$. Let A_0 and B_0 be functions of type $(\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ defined as follows.*

$$\begin{array}{llll} X_0: & (\mathbb{R}^k)^n & \to & (\mathbb{R}^k)^n \\ & x & \mapsto & q^X(x) \end{array}$$

where X represents A or B, $q^X(x) = \big(q^X(x)_1, \ldots, q^X(x)_n\big)$,
$\forall i \in \{1, \ldots, n\}, q^X(x)_i = (q_1^X(x)_i, \ldots, q_k^X(x)_i)$,
$\forall l \in \{1, \ldots, k\}, q_l^X(x) = \big(q_l^X(x)_1, \ldots, q_l^X(x)_n\big)$
and $\forall i \in \{1, \ldots, n\}$ and $\forall l \in \{1, \ldots, k\}, q_l^X(x)_i \in [0, 1]$.

[36] Krishna, V.: Auction Theory, Academic Press, 2009, p. 204.
Please note that also the examples of multi-unit auction formats introduced in Section 2.3 are modeled in this way.

Furthermore, let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be functions of type $(\mathbb{R}^k)^n \to \mathbb{R}$. For all $i \in \{1, \ldots, n\}$, let the function u_i be defined as follows.

$$\begin{array}{rccc} u_i: & \mathbb{R}^k & \to & \mathbb{R}^k \\ & x & \mapsto & x \end{array}$$

For X representing A or B and for every natural number i between 1 and n, let us assume that the functions e_i^X and a_i^X below are well-defined.

$$\begin{array}{rlcl} e_i^X: & \mathbb{R}^k & \to & \mathbb{R} \\ & v & \mapsto & E(X_i(V[i \to v])) \end{array}$$

$$\begin{array}{rlcl} a_i^X: & (\mathbb{R}^k)^2 & \to & \mathbb{R}^k \\ & x, y & \mapsto & E(\langle u_i(V[i \to x]); q^X(V[i \to y])_i \rangle) \\ & & & \\ & & & \text{where } V[i \to v] := (V_1, \ldots, V_{i-1}, v, V_{i+1}, \ldots, V_n), \\ & & & \forall i \in \{1, \ldots, n\}, u_i = (u_{i,1}, \ldots, u_{i,k}) \\ & & & \text{and } \langle u_i(V[i \to x]); q^X(V[i \to y])_i \rangle := \sum_{l=1}^k u_{i,l}(V[i \to x]) q_l^X(V[i \to y])_i \end{array}$$

We assume the following.

1. *For all i between 1 and n, the function a_i^X is differentiable.*
2. *For all i between 1 and n and for all distinct $x, y \in \mathbb{R}^k$*

   $$a_i^X(x, y) - e_i^X(y) < a_i^X(x, x) - e_i^X(x).$$

3. *The random vectors $V_1, \ldots, V_n$ are mutually independent.*
4. *For all i between 1 and n, let f_i be a density function for V_i.*
5. *For all $i \in \{1, \ldots, n\}$ we have $a_i^A = a_i^B$.*
6. *Let $(v_i)_{i \in \{1, \ldots, n\}}$ be a family of k-vectors, such that $\sum_{i=1}^n e_i^A(v_i) = \sum_{i=1}^n e_i^B(v_i)$.*

The following can be deduced from the assumptions above.

$$E(\sum_{i=1}^n A_i(V)) = E(\sum_{i=1}^n B_i(V))$$

As we can see, Krishna and Perry only work with the value v_i to express the valuation of bidder i. Thus, the function u_i is just the identity function. Furthermore, they allow for all i between 1 and n, a density function f_i for the random vector V_i.

4.3 Generalization of the multi-unit revenue equivalence theorem

Now let us prove the generalized version of the theorem. In fact, we are able to prove the same results as in the single-unit version of the theorem, there are only slightly changes in the way of proof.

Theorem 4.3. *Let n and k be two natural numbers. For all i between 1 and n, let $V_i = (V_i^1, \dots, V_i^k)$ be a random vector and let $V := (V_1, \dots, V_n)$. Let A_0 and B_0 be functions of type $(\mathbb{R}^k)^n \to (\mathbb{R}^k)^n$ defined as follows.*

$$\begin{array}{llll} X_0: & (\mathbb{R}^k)^n & \to & (\mathbb{R}^k)^n \\ & x & \mapsto & q^X(x) \\ & & & \\ & & & \text{where } X \text{ represents } A \text{ or } B,\ q^X(x) = \big(q^X(x)_1, \dots, q^X(x)_n\big), \\ & & & \forall i \in \{1, \dots, n\}, q^X(x)_i = (q_1^X(x)_i, \dots, q_k^X(x)_i), \\ & & & \forall l \in \{1, \dots, k\}, q_l^X(x) = \big(q_l^X(x)_1, \dots, q_l^X(x)_n\big) \\ & & & \text{and } \forall i \in \{1, \dots, n\} \text{ and } \forall l \in \{1, \dots, k\}, q_l^X(x)_i \in [0,1]. \end{array}$$

Furthermore, let $A_1, \dots, A_n$ and $B_1, \dots, B_n$ be functions of type $(\mathbb{R}^k)^n \to \mathbb{R}$ and let $u_1, \dots, u_n$ be functions of type $(\mathbb{R}^k)^n \to \mathbb{R}^k$. For X representing A or B and for every natural number i between 1 and n, let us assume that the functions e_i^X, $p_{i,l}^X$ and a_i^X below are well-defined.

$$\begin{array}{llll} e_i^X: & \mathbb{R}^k & \to & \mathbb{R} \\ & v & \mapsto & E(X_i(V[i \to v])) \end{array}$$

$$\begin{array}{llll} p_{i,l}^X: & \mathbb{R}^k & \to & [0,1] \\ & v & \mapsto & E(q_l^X(V[i \to v])_i), \quad \forall l \in \{1, \dots, k\} \end{array}$$

$$\begin{array}{llll} a_i^X: & (\mathbb{R}^k)^2 & \to & \mathbb{R} \\ & x, y & \mapsto & E(\langle u_i(V[i \to x]); q^X(V[i \to y])_i \rangle) \\ & & & \\ & & & \text{where } V[i \to v] := (V_1, \dots, V_{i-1}, v, V_{i+1}, \dots, V_n), \\ & & & \forall i \in \{1, \dots, n\}, u_i = (u_{i,1}, \dots, u_{i,k}) \\ & & & \text{and } \langle u_i(V[i \to x]); q^X(V[i \to y])_i \rangle := \sum_{l=1}^k u_{i,l}(V[i \to x]) q_l^X(V[i \to y])_i \end{array}$$

We assume the following.

1. *For all i between 1 and n, the function a_i^X is differentiable.*
2. *For all i between 1 and n and for all distinct $x, y \in \mathbb{R}^k$*
$$a_i^X(x, y) - e_i^X(y) < a_i^X(x, x) - e_i^X(x).$$
3. *The random vectors $V_1, \dots, V_n$ are mutually independent.*

4. *For all i between 1 and n, let f_i be a density function for V_i. Let $J_1, \dots, J_s$ be a partition of $\{1, \dots, n\}$ complying with the following. $\forall i, j \in \{1, \dots, n\}$, $f_i = f_j \Rightarrow \exists\, r \in \{1, \dots, s\}, i, j \in J_r$. Also, for all r between 1 and s, let the function f^r be defined as follows. $\forall i \in \{1, \dots, n\}, i \in J_r \Rightarrow f^r := f_i$.*
5. *For all $r \in \{1, \dots, s\}$ we have $\sum_{i \in J_r} p_i^A = \sum_{i \in J_r} p_i^B$, $\sum_{i \in J_r} a_i^A = \sum_{i \in J_r} a_i^B$ and $\sum_{i \in J_r} \frac{\partial a_i^A}{\partial x_1} = \sum_{i \in J_r} \frac{\partial a_i^B}{\partial x_1}$*
6. *Let $(v^r)_{r \in \{1,\dots,s\}}$ be a family of k-vectors, such that $\sum_{r=1}^{s} \sum_{i \in J_r} e_i^A(v^r) = \sum_{r=1}^{s} \sum_{i \in J_r} e_i^B(v^r)$.*

The following can be deduced from the assumptions above.

a) $E(\sum_{i=1}^{n} A_i(V)) = E(\sum_{i=1}^{n} B_i(V))$

b) $\forall l \in \{1, \dots, k\}, \sum_{i=1}^{n} E(q_l^A(V)_i) = \sum_{i=1}^{n} E(q_l^B(V)_i)$

c) $\sum_{i=1}^{n} e_i^A = \sum_{i=1}^{n} e_i^B$

d) *If there exists $v \in \mathbb{R}^k$ and $r \in \{1, \dots, s\}$ such that $\sum_{i \in J_r} e_i^A(v) = \sum_{i \in J_r} e_i^B(v)$, then $\sum_{i \in J_r} e_i^A = \sum_{i \in J_r} e_i^B$.*

Proof. First notice that we can apply Lemma 4.1 for all i between 1 and n for the functions a_i^X and e_i^X, when we replace the functions f and g with a_i^X and e_i^X respectively, since by assumption 1, a_i^X is differentiable and since the types of a_i^X and e_i^X do not restrict the application of the lemma.

a) Since assumption 2 is equivalent to assertion 1 of Lemma 4.1, we can use assertion 2 of the lemma. Then the following holds for all $i \in \{1, \dots, n\}$ and for all $x, y \in \mathbb{R}^k$, where $u^* = x + u(y - x)$.

$$e_i^X(x) = e_i^X(y) - a_i^X(y, y) + a_i^X(x, x) + \int_0^1 \frac{\partial a_i^X}{\partial x_1}(u^*, u^*)du \qquad (10)$$

By basic probability theory $E(\sum_{i=1}^{n} X_i(V)) = \sum_{i=1}^{n} E(X_i(V))$. For all $i \in \{1, \dots, n\}$, we use the density function f_i of V_i according to assumption 4. So $E(X_i(V)) = E(X_i(V[i \to V_i])) = \int_{\mathbb{R}^k} e_i^X(x) f_i(x)dx$ by independence in assumption 3. Therefore, using equation (10), the following holds for all $y \in \mathbb{R}^k$, where $u^* = x + u(y - x)$.

$$E(\sum_{i=1}^{n} X_i(V)) = \sum_{i=1}^{n} \int_{\mathbb{R}^k} \left(e_i^X(y) - a_i^X(y, y) + a_i^X(x, x) + \int_0^1 \frac{\partial a_i^X}{\partial x_1}(u^*, u^*)du \right) f_i(x)dx$$

We now use the partition of the set $\{1, \dots, n\}$ (see assumption 4). Then the equation above yields for all $y \in \mathbb{R}^k$.

$$\begin{aligned} E(\sum_{i=1}^{n} A_i(V)) &= \sum_{i=1}^{n} \int_{\mathbb{R}^k} \left(e_i^A(y) - a_i^A(y, y) + a_i^A(x, x) + \int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*, u^*)du \right) f_i(x)dx \\ &= \sum_{r=1}^{s} \sum_{i \in J_r} \int_{\mathbb{R}^k} \left(e_i^A(y) - a_i^A(y, y) + a_i^A(x, x) + \int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*, u^*)du \right) f_i(x)dx \end{aligned}$$

Since the equation above holds for all $y \in \mathbb{R}^k$, we use the family $(v^r)_{r\in\{1,\dots,s\}}$ (see assumption 6). Now we conclude using also assumptions 4 and 5, linearity of integration and summation and the fact that $\forall i \in \{1,\dots,n\}, \int_{\mathbb{R}^k} f_i(x)dx = 1$.

$$
\begin{aligned}
E(\sum_{i=1}^{n} A_i(V)) &= \sum_{r=1}^{s}\sum_{i\in J_r}\int_{\mathbb{R}^k}\left(e_i^A(v^r) - a_i^A(v^r,v^r) + a_i^A(x,x) + \int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*,u^*)du\right) f^r(x)dx \\
&= \sum_{r=1}^{s}\sum_{i\in J_r} e_i^A(v^r) - \sum_{r=1}^{s}\sum_{i\in J_r} a_i^A(v^r,v^r) \\
&\quad + \sum_{r=1}^{s}\sum_{i\in J_r}\int_{\mathbb{R}^k}\left(a_i^A(x,x) + \int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*,u^*)du\right) f^r(x)dx \\
&= \sum_{r=1}^{s}\sum_{i\in J_r} e_i^A(v^r) - \sum_{r=1}^{s}\sum_{i\in J_r} a_i^A(v^r,v^r) \\
&\quad + \sum_{r=1}^{s}\int_{\mathbb{R}^k}\left(\sum_{i\in J_r} a_i^A(x,x) + \int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*,u^*)du\right) f^r(x)dx \\
&= \sum_{r=1}^{s}\sum_{i\in J_r} e_i^B(v^r) - \sum_{r=1}^{s}\sum_{i\in J_r} a_i^B(v^r,v^r) \\
&\quad + \sum_{r=1}^{s}\int_{\mathbb{R}^k}\left(\sum_{i\in J_r} a_i^B(x,x) + \int_0^1 \frac{\partial a_i^B}{\partial x_1}(u^*,u^*)du\right) f^r(x)dx \\
&= E(\sum_{i=1}^{n} B_i(V))
\end{aligned}
$$

b) By assumption 5 we obtain for all l between 1 and k and for all $r \in \{1,\dots,s\}$ that $\sum_{i\in J_r} p_{i,l}^A = \sum_{i\in J_r} p_{i,l}^B$. Using this assertion we conclude for all $l \in \{1,\dots,k\}$.

$$
\begin{aligned}
\sum_{i=1}^{n} E(q_l^A(V)_i) &= \sum_{i=1}^{n} E(q_l^A(V[i \to V_i])_i) = \sum_{i=1}^{n}\int_{\mathbb{R}^k} p_{i,l}^A(x) f_i(x)dx \\
&= \sum_{r=1}^{k}\sum_{i\in J_r}\int_{\mathbb{R}^k} p_{i,l}^A(x) f^r(x)dx = \sum_{r=1}^{s}\int_{\mathbb{R}^k}\sum_{i\in J_r} p_{i,l}^A(x) f^r(x)dx \\
&= \sum_{r=1}^{s}\int_{\mathbb{R}^k}\sum_{i\in J_r} p_{i,l}^B(x) f^r(x)dx \sum_{i=1}^{n} E(q_l^B(V)_i)
\end{aligned}
$$

c) We use the family $(v^r)_{r\in\{1,\dots,s\}}$ (see assumption 6) and assumption 5. Then, by equation (10), we obtain for all $x \in \mathbb{R}^k$, where $u^* = x + u(v^r - x)$.

$$\sum_{r=1}^{s}\sum_{i\in J_r} e_i^X(x) = \sum_{r=1}^{s}\sum_{i\in J_r} e_i^X(v^r) - \sum_{r=1}^{s}\sum_{i\in J_r} a_i^X(v^r, v^r) + \sum_{r=1}^{s}\sum_{i\in J_r} a_i^X(x, x) + \sum_{r=1}^{s}\int_0^1 \frac{\partial a_i^X}{\partial x_1}(u^*, u^*)du$$

Thus we conclude.

$$\begin{aligned}\sum_{r=1}^{s}\sum_{i\in J_r} e_i^A(x) &= \sum_{r=1}^{s}\sum_{i\in J_r} e_i^A(v^r) - \sum_{r=1}^{s}\sum_{i\in J_r} a_i^A(v^r, v^r) \\ &\quad + \sum_{r=1}^{s}\sum_{i\in J_r} a_i^A(x, x) + \sum_{r=1}^{s}\int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*, u^*)du \\ &= \sum_{r=1}^{s}\sum_{i\in J_r} e_i^B(v^r) - \sum_{r=1}^{s}\sum_{i\in J_r} a_i^B(v^r, v^r) \\ &\quad + \sum_{r=1}^{s}\sum_{i\in J_r} a_i^B(x, x) + \sum_{r=1}^{s}\int_0^1 \frac{\partial a_i^B}{\partial x_1}(u^*, u^*)du \\ &= \sum_{r=1}^{s}\sum_{i\in J_r} e_i^B(x)\end{aligned}$$

Since this is true for all $x \in \mathbb{R}^k$ we obtain

$$\sum_{i=1}^{n} e_i^A = \sum_{r=1}^{s}\sum_{i\in J_r} e_i^A = \sum_{r=1}^{s}\sum_{i\in J_r} e_i^B = \sum_{i=1}^{n} e_i^B.$$

d) Let $r \in \{1, \ldots, s\}$ and $v \in \mathbb{R}^k$ such that

$$\sum_{i\in J_r} e_i^A(v) = \sum_{i\in J_r} e_i^B(v) \tag{11}$$

Analogous to the proof of *c*), we use equation (10) and obtain for all $x \in \mathbb{R}^k$, where $u^* = x + u(v - x)$.

$$\sum_{i\in J_r} e_i^X(x) = \sum_{i\in J_r} e_i^X(v) - \sum_{i\in J_r} a_i^X(v, v) + \sum_{i\in J_r} a_i^X(x, x) + \int_0^1 \frac{\partial a_i^X}{\partial x_1}(u^*, u^*)du$$

Now by using assumption 5 and equation (11) we conclude.

$$\begin{aligned}\sum_{i\in J_r} e_i^A(x) &= \sum_{i\in J_r} e_i^A(v) - \sum_{i\in J_r} a_i^A(v, v) + \sum_{i\in J_r} a_i^A(x, x) + \int_0^1 \frac{\partial a_i^A}{\partial x_1}(u^*, u^*)du \\ &= \sum_{i\in J_r} e_i^B(v) - \sum_{i\in J_r} a_i^B(v, v) + \sum_{i\in J_r} a_i^B(x, x) + \int_0^1 \frac{\partial a_i^B}{\partial x_1}(u^*, u^*)du \\ &= \sum_{i\in J_r} e_i^B(x)\end{aligned}$$

Since this is true for all $x \in \mathbb{R}^k$ we obtain

$$\sum_{i \in J_r} e_i^A = \sum_{i \in J_r} e_i^B.$$

$\square$

Let us first clarify our interpretation of the "value-vector" $v_i = (v_i^1, \ldots, v_i^k)$ of bidder i when there are k identical objects to be auctioned-off. By v_i^1 we identify bidder i's value for obtaining one object, v_i^2 represents his value for two objects and v_i^l represents bidder i's valuation for obtaining exactly l objects. We do not require any assumptions regarding the values v_i^l. Using this interpretation we remark that the function $q^l(\cdot)_i$ represents the probability that bidder i will get exactly l objects.

As mentioned above, we managed to follow nearly the same way of proof as in the generalized version of the single-unit revenue equivalence theorem. In addition, all our assumptions are just the multi-dimensional extensions of the single-unit version. We again allow for all bidders i any function u_i, expressing their value/utility for the object(s). Also, by assumption 4 we introduce a partition of the set $\{1, \ldots, n\}$ complying with the appearance of the density functions of the V_i's. Since we now have multiple objects, the formula of the expected payoff of the bidders (assumption 2) has been adjusted.

5 Optimal auctions

5.1 Framework and preliminaries for the optimal auction

The revenue equivalence theorem is a powerful result that shows us which auction formats yield the same expected payoff for the seller when they have some properties in common. We now want to analyse which class of auction formats ensure the highest expected payoff for the seller, i.e. we will discuss the requirements for an optimal auction. After discussing the framework we will first revisit Myerson's result of an optimal auction and then characterize the optimal auction in our new formalism leading to a different result.

First, let $v_0 \in \mathbb{R}$ define the value of the object to be auctioned-off for the seller. We also will use a function u_0 of type $\mathbb{R}^{n+1} \to \mathbb{R}$ to express that the sellers value may also be influenced by the values of the bidders. We will characterize the expected payoff U_0 of the seller as the sum of all expected payments by the bidders and his own valuation in the case the object remains unsold after the auction.

5.2 Optimal auction by Myerson

In order to be able to characterize Myerson's version of the optimal auction we will again use the "revision effect" functions, defined in Section 3.4, Definition 3.4 and the interval $[\alpha, \beta]$ (in addition, we define the interval from which the seller chooses his value v_0 for the object by $[\alpha_0, \beta_0]$). Furthermore we introduce parts of his formalism regarding the expected payoff of the bidders. In the previous sections we explained that we can describe the expected payoff by our functions a and e by $a(x,x) - e(x)$. Let now $U(x)$ identify this expected payoff, i.e. let $n \in \mathbb{R}$, then

$$\forall i \in \{1, \ldots, n\}, \forall x \in \mathbb{R}, U_i(x) := a_i(x,x) - e_i(x). \tag{12}$$

Myerson derives a new equivalence for the functions $U_i(x)$ we will use for the proof.

Theorem 5.1. *Let n be a natural number and $V = (V_1, \ldots, V_n)$ be a vector consisting of n real-valued random variables. For all i between 1 and n, let the functions O_i be of type $\mathbb{R}^n \to \mathbb{R}$ and let the functions u_i and u_0 be defined as follows.*

$$\begin{array}{llll} u_i: & [\alpha, \beta] & \to & \mathbb{R} \\ & v & \mapsto & v_i + \sum_{j \in \{1,\ldots,n\}, j \neq i} \tau_j(v_j) \end{array}$$

$$\begin{array}{llll} u_0: & [\alpha, \beta]_0 & \to & \mathbb{R} \\ & v_0, v & \mapsto & v_0 + \sum_{j \in \{1,\ldots,n\}} \tau_j(v_j) \end{array}$$

where $v = (v_1, \ldots, v_n)$ and $[\alpha, \beta]_0 := [\alpha_0, \beta_0] \times [\alpha, \beta]$.

Let the functions O_0, a_0 and U_0 be defined as follows.

$$\begin{array}{llll} O_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & q^O(x) \\ & & & \\ & & & \textit{where } q^O(x) = \big(q^O(x)_1, \dots, q^O(x)_n\big), \\ & & & \forall i \in \{1, \dots, n\}, 0 \le q^O(x)_i \textit{ and } \sum_{i=1}^n q^O(x)_i \le 1. \end{array}$$

$$\begin{array}{llll} a_0: & [\alpha_0, \beta_0] & \to & \mathbb{R} \\ & v & \mapsto & E(u_0(v, V)) \end{array}$$

$$\begin{array}{llll} U_0: & \mathbb{R} & \to & \mathbb{R} \\ & v & \mapsto & a_0(v)\big(1 - \sum_{i=1}^n E(q^O(V)_i)\big) + \sum_{i=1}^n E(O_i(V)) \end{array}$$

Furthermore, for all $i \in \{1, \dots, n\}$, we define the functions e_i^O, p_i^O and a_i^O in the same way as in the revenue equivalence theorem.

$$\begin{array}{llll} e_i^O: & [\alpha_i, \beta_i] & \to & \mathbb{R} \\ & v & \mapsto & E(O_i(V[i \to v])) \end{array}$$

$$\begin{array}{llll} p_i^O: & [\alpha_i, \beta_i] & \to & [0, 1] \\ & v & \mapsto & E(q^O(V[i \to v]_i)) \end{array}$$

$$\begin{array}{llll} a_i^O: & ([\alpha_i, \beta_i])^2 & \to & \mathbb{R} \\ & x, y & \mapsto & E(u_i(V[i \to x]) \cdot q^O(V[i \to y]_i)) \\ & & & \\ & & & \textit{where } V[i \to v] := (V_1, \dots, V_{i-1}, v, V_{i+1}, \dots, V_n) \end{array}$$

Finally, for all i between 1 and n, let the function U_i be defined as follows.

$$\begin{array}{llll} U_i: & [\alpha_i, \beta_i] & \to & \mathbb{R} \\ & v & \mapsto & U_i(\alpha_i) + \int_{\alpha_i}^v p_i^O(x)dx \end{array}$$

We assume the following.

1. *The random variables $V_1, \dots, V_n$ are mutually independent.*
2. *For all i between 1 and n we have*

$$O_i(v) = q^O(v)_i u_i(v) - \int_{\alpha_i}^{v_i} q^O(v_{-i}, s_i)_i ds_i$$

   where $v \in [\alpha, \beta]$, $v = (v_1, \dots, v_n)$ and $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.

3. *For all i between 1 and n we have $0 \le U_i(\alpha_i)$.*
4. *For all i between 1 and n, let f_i be a density function and F_i be a distribution function for V_i. Furthermore, let $f := \prod_{i=1}^n f_i$ and $f_{-i} = \prod_{j=1, j \ne i}^n f_j$.*

Let $v_0 \in \mathbb{R}$. If now the function O_0 maximizes

$$\int_{[\alpha,\beta]} \Big(\sum_{i=1}^{n} \big(v_i - v_0 - \tau_i(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)}\big) q^O(v)_i\Big) f(v) dv,$$

where $v = (v_1, \ldots, v_n)$, then the function U_0 is maximal subject to the assumptions above.

Proof. By using the notation above for the densities and by the definition of the expectation we obtain for the function U_0.

$$\begin{aligned} U_0(v_0) &= a_0(v_0)\big(1 - \sum_{i=1}^{n} E(q^O(V)_i)\big) + \sum_{i=1}^{n} E(O_i(V)) \\ &= a_0(v_0)\big(1 - \sum_{i=1}^{n} \int_{[\alpha,\beta]} q^O(v)_i f(v) dv\big) + \sum_{i=1}^{n} \int_{[\alpha,\beta]} O_i(v) f(v) dv \end{aligned}$$

By rearranging the equation above we obtain.

$$\begin{aligned} U_0(v_0) = a_0(v_0) &+ \sum_{i=1}^{n} \int_{[\alpha,\beta]} q^O(v)_i \big(u_i(v) - u_0(v_0, v)\big) f(v) dv \\ &+ \sum_{i=1}^{n} \int_{[\alpha,\beta]} O_i(v) - q^O(v)_i u_i(v) f(v) dv \end{aligned} \tag{13}$$

By using the characterization for the functions U_i we made in the preliminaries (see equation (12)) we obtain for all $i \in \{1, \ldots, n\}$.

$$\begin{aligned} \int_{[\alpha,\beta]} O_i(v) - q^O(v)_i u_i(v) f(v) dv &= -\int_{\alpha_i}^{\beta_i} U_i(v_i) f_i(v_i) dv_i \\ &= -\int_{\alpha_i}^{\beta_i} \big(U_i(\alpha_i) + \int_{\alpha_i}^{v_i} p_i^O(x) dx\big) f_i(v_i) dv_i \\ &= -U_i(\alpha_i) - \int_{\alpha_i}^{\beta_i} \int_{x}^{\beta_i} f_i(v_i) p_i^O(x) dv_i dx \\ &= -U_i(\alpha_i) - \int_{\alpha_i}^{\beta_i} (1 - F_i(x)) p_i^O(x) dx \\ &= -U_i(\alpha_i) - \int_{[\alpha,\beta]} (1 - F_i(v)) q^O(v)_i f_{-i}(v_{-i}) dv \end{aligned} \tag{14}$$

By the definitions of the functions u_i and u_0 we obtain.

$$\forall i \in \{1, \ldots, n\}, u_i(v) - u_0(v_0, v) = v_i - v_0 - \tau_i(v_i) \tag{15}$$

Now we conclude by equations (14) and (15).

$$\begin{aligned} U_0(v_0) = &\int_{[\alpha,\beta]} \Big(\sum_{i=1}^{n} \big(v_i - v_0 - \tau_i(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)}\big) q^O(v)_i\Big) f(v) dv \\ &+ a_0(v_0) - \sum_{i=1}^{n} U_i(\alpha_i) \end{aligned} \tag{16}$$

Finally, we obtain for all $i \in \{1, \ldots, n\}$ by using also assumption 3 and the definition of U_i in the beginning of the theorem.

$$0 \leq U_i(\alpha_i) = \int_{[\alpha,\beta]_{-i}} \big(u_i(v)q^O(v)_i - \int_{\alpha_i}^{v_i} q^O(v_{-i}, s_i)_i ds_i - O_i(v)\big) f_{-i}(v_{-i}) dv_{-i}$$

$$\text{where } [\alpha,\beta]_{-i} = [\alpha_1, \beta_1] \times \cdots \times [\alpha_{i-1}, \beta_{i-1}] \times [\alpha_{i+1}, \beta_{i+1}] \times \cdots \times [\alpha_n, \beta_n].$$

Now we conclude that by using assumption 2 we get $\sum_{i=1}^n U_i(\alpha_i) = 0$, which is the best possible value in equation (16). Finally, since $a_0(v_0)$ is a constant, independent of the functions O_0 and O_i we obtain that U_0 is maximal if O_0 maximizes

$$\int_{[\alpha,\beta]} \Big(\sum_{i=1}^n \big(v_i - v_0 - \tau_i(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)}\big) q^O(v)_i\Big) f(v) dv.$$

□

5.3 Construction of Myerson's optimal auction

We will characterize Myerson's construction of an optimal auction. Here, we will present his construction for the "regular case"[37].

Definition 5.2. *The problem to find an optimal auction is regular, if for all i between 1 and n, the function*

$$c_i(v_i) = v_i - \tau_i(v_i) - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

is strictly increasing for all $v_i \in [\alpha_i, \beta_i]$.

We now consider the following auction format: The seller keeps the object if $\max_{i\in\{1,\ldots,n\}} (c_i(v_i)) < v_0$, otherwise he sells the object to the bidder with the highest $c_i(v_i)$. In case of a draw,

$$\exists i, j \in \{1, \ldots, n\}, i \neq j, v_0 \leq c_i(v_i) = c_j(v_j) = \max_{k\in\{1,\ldots,n\}} (c_k(v_k)),$$

some arbitrary rule may be used. Thus, in this auction format,

$$\forall i \in \{1, \ldots, n\}, 0 < q^O(v)_i \Rightarrow v_0 \leq c_i(v_i) = \max_{j\in\{1,\ldots,n\}} (c_j(v_j)).$$

For all $v \in [\alpha, \beta]$, this considered rule maximizes the sum $\sum_{i=1}^n (c_i(v_i) - v_0) q^O(v)_i$, subject to $\sum_{i=1}^n q^O(v)_i \leq 1$ and $\forall i \in \{1, \ldots, n\}, 0 \leq q^O(v)_i$. Now we consider again the function O_i as defined in assumption 2. Before deriving the payments by the bidders we introduce for all i between 1 and n the function z_i as follows,

$$z_i(v_{-i}) = \inf \{s_i | v_0 \leq c_i(s_i) \text{ and } \forall j \in \{1, \ldots, n\}, i \neq j, c_j(v_j) \leq c_i(v_i)\}$$

[37]Beside the regular case, Myerson analyses also a general case for an optimal auction.

where again $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. So, $z_i(v_{-i})$ is the infimum of all winning bids for bidder i. We thus obtain for all $i \in \{1, \ldots, n\}$.

$$q^O(v_{-i}, s_i)_i = \begin{cases} 1 & \text{if} \quad s_i > z_i(v_{-i}) \\ 0 & \text{if} \quad s_i < z_i(v_{-i}) \end{cases}$$

By the equation above we conclude.

$$\int_{\alpha_i}^{v_i} q^O(v_{-i}, s_i) ds_i = \begin{cases} v_i - z_i(v_{-i}) & \text{if} \quad v_i > z_i(v_{-i}) \\ 0 & \text{if} \quad v_i < z_i(v_{-i}) \end{cases}$$

Finally, we can compute the functions O_i for all $i \in \{1, \ldots, n\}$.

$$O_i(v) = \begin{cases} z_i(v_{-i}) + \sum_{j \in \{1,\ldots,n\}_{j \neq i}} \tau_j(v_j) & \text{if} \quad q^O(v)_i = 1 \\ 0 & \text{if} \quad q^O(v)_i = 0 \end{cases}$$

By this payment function we see that in Myerson's optimal auction bidder i only pays when he wins the auction. In this case he pays an amount equal to $u_i(v_{-i}, z_i(v_{-i}))$. If we now assume that all the revision effect functions are identically zero and if for all $i, j \in \{1, \ldots, n\}, \alpha_i = \alpha_j, \beta_i = \beta_j, f_i = f_j$, then we obtain.

$$\forall i \in \{1, \ldots, n\}, z_i(v_{-i}) = \max\{c_i^{-1}(v_0), \max_{j \in \{1,\ldots,n\}, j \neq i} v_j\}$$

By this construction, Myersons optimal auction becomes a second price auction with a reserve price $c_i^{-1}(v_0)$ (by the regularity assumption the function c_i is invertible and by the restrictions above, $\forall i, j \in \{1, \ldots, n\}, c_i = c_j$).

5.4 New formalism for an optimal auction

We now return to the formalism introduced in Section 3. We will introduce other assumptions and requirements for an optimal auction and will see that the new assumptions lead to a different solution.

Theorem 5.3. *Let n be a natural number and $V = (V_1, \ldots, V_n)$ be a vector consisting of n real-valued random variables. For all i between 1 and n, let the functions O_i and u_i be of type $\mathbb{R}^n \to \mathbb{R}$ and let the function u_0 be of type $\mathbb{R}^{n+1} \to \mathbb{R}$. Let the functions O_0, a_0 and U_0 be defined as follows.*

$$\begin{array}{rlll} O_0: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & q^O(x) \end{array}$$

$$\text{where } q^O(x) = \big(q^O(x)_1, \ldots, q^O(x)_n\big), \\ \forall i \in \{1, \ldots, n\}, 0 \leq q^O(x)_i \text{ and } \textstyle\sum_{i=1}^n q^O(x)_i \leq 1.$$

$$\begin{array}{rlll} a_0: & \mathbb{R} & \to & \mathbb{R} \\ & v & \mapsto & E(u_0(v, V) \end{array}$$

$$\begin{array}{rlll} U_0: & \mathbb{R} & \to & \mathbb{R} \\ & v & \mapsto & a_0(v)\big(1 - \sum_{i=1}^n E(q^O(V)_i)\big) + \sum_{i=1}^n E(O_i(V)) \end{array}$$

Furthermore, for all $i \in \{1, \dots, n\}$, we define the functions e_i^O and a_i^O in the same way as in the revenue equivalence theorem.

$$\begin{array}{rlcl} e_i^O: & \mathbb{R} & \to & \mathbb{R} \\ & v & \mapsto & E(O_i(V[i \to v])) \end{array}$$

$$\begin{array}{rlcl} a_i^O: & \mathbb{R}^2 & \to & \mathbb{R} \\ & x, y & \mapsto & E(u_i(V[i \to x]) \cdot q^O(V[i \to y]_i)) \end{array}$$

$$\text{where } V[i \to v] := (V_1, \dots, V_{i-1}, v, V_{i+1}, \dots, V_n)$$

We assume the following.

1. *The random variables $V_1, \dots, V_n$ are mutually independent.*
2. *For all i between 1 and n we have $O_i = q_i^O u_i$.*
3. *For all i between 1 and n, let f_i be a density function for V_i. Let $J_1, \dots, J_k$ be a partition of $\{1, \dots, n\}$ complying with the following.* $\forall i, j \in \{1, \dots, n\}, f_i = f_j \Rightarrow \exists l \in \{1, \dots, k\}, i, j \in J_l$. *Also, for all l between 1 and k, let the function f^l be defined as follows.* $\forall i \in \{1, \dots, n\}, i \in J_l \Rightarrow f^l := f_i$. *Furthermore, let $f := \prod_{l=1}^k f^l$.*
4. $\forall y \in \mathbb{R}, 0 \leq \sum_{l=1}^k \sum_{i \in J_l} \int_{\mathbb{R}} a_i^O(y, y) - e_i^O(y) - \int_x^y \frac{\partial a_i^O}{\partial x_1}(z, z) dz f^l(x) dx.$
5. *For all i between 1 and n, the function a_i^O is differentiable.*
6. *For all i between 1 and n and for all distinct real numbers x and y,*

$$a_i^O(x, y) - e_i^O(y) < a_i^O(x, x) - e_i^O(x)$$

Let $v_0 \in \mathbb{R}$. If now the function O_0 maximizes

$$\sum_{l=1}^k \sum_{i \in J_l} \int_{\mathbb{R}^n} q^O(v)_i (u_i(v) - u_0(v_0, v)) f(v) dv,$$

where $v = (v_1, \dots, v_n)$, then the function U_0 is maximal subject to the assumptions above.

Proof. By using the notation above for the densities and by the definition of the expectation we obtain for the function U_0.

$$\begin{aligned} U_0(v_0) &= a_0(v_0)\big(1 - \sum_{l=1}^k \sum_{i \in J_l} E(q^O(V)_i)\big) + \sum_{l=1}^k \sum_{i \in J_l} E(O_i(V)) \\ &= a_0(v_0)\big(1 - \sum_{l=1}^k \sum_{i \in J_l} \int_{\mathbb{R}^n} q^O(v)_i f(v) dv\big) + \sum_{l=1}^k \sum_{i \in J_l} \int_{\mathbb{R}^n} O_i(v) f(v) dv \end{aligned}$$

By rearranging the equation above we obtain.

$$\begin{aligned} U_0(v_0) = a_0(v_0) &+ \sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}^n} q^O(v)_i \big(u_i(v) - u_0(v_0, v)\big) f(v) dv \\ &+ \sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}^n} O_i(v) - q^O(v)_i u_i(v) f(v) dv \end{aligned} \tag{17}$$

Since by assumption 2, for all $i \in \{1, \ldots, n\}$, $O_i = q_i^O u_i$ we obtain that the last addend in the equation above equals zero. We now have to show that this is indeed the best possible value for the addend. First, for all i between 1 and n, we replace the functions O_i, q_i^O and u_i by the functions e_i^O and a_i^O in the following way.

$$\begin{aligned} \sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}^n} O_i(v) - q^O(v)_i u_i(v) f(v) dv &= \sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} e_i^O(v_i) - a_i^O(v_i, v_i) f^l(v_i) dv_i \\ &= -\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} a_i^O(v_i, v_i) - e_i^O(v_i) f^l(v_i) dv_i \end{aligned} \tag{18}$$

where $v = (v_1, \ldots, v_n)$.

Since for all $i \in \{1, \ldots, n\}$, the function a_i^O is differentiable by assumption 5 and since assumption 6 is equivalent to assertion 1 of Lemma 3.1, we can use assertion 2 of the lemma. In the same way as in the proof of the revenue equivalence theorem, the following holds for all $i \in \{1, \ldots, n\}$ and for all real numbers x and y.

$$a_i^O(x, x) - e_i^O(x) = a_i^O(y, y) - e_i^O(y) - \int_x^y \frac{\partial a_i^O}{\partial x_1}(z, z) dz \tag{19}$$

We apply equation (19) in equation (18) and obtain for all $y \in \mathbb{R}$.

$$-\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} a_i^O(v_i, v_i) - e_i^O(v_i) f^l(v_i) dv_i = -\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} \big(a_i^O(y, y) - e_i^O(y) - \int_{v_i}^y \frac{\partial a_i^O}{\partial x_1}(z, z) dz\big) f^l(v_i) dv_i$$

By using assumption 4 we conclude.

$$-\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} a_i^O(v_i, v_i) - e_i^O(v_i) f^l(v_i) dv_i = -\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} \big(a_i^O(y, y) - e_i^O(y) - \int_{v_i}^y \frac{\partial a_i^O}{\partial x_1}(z, z) dz\big) f^l(v_i) dv_i \leq 0$$

Thus, by applying equation (19) in equation (17) we obtain for all $y \in \mathbb{R}$.

$$\begin{aligned} U_0(v_0) = a_0(v_0) &+ \sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}^n} q^O(v)_i \big(u_i(v) - u_0(v_0, v)\big) f(v) dv \\ &- \sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}} a_i^O(y, y) - e_i^O(y) - \int_{v_i}^y \frac{\partial a_i^O}{\partial x_1}(z, z) dz f^l(v_i) dv_i \end{aligned}$$

We can see now that 0 is the best possible value for the last addend in the equation above. Furthermore, $a_0(v_0) \in \mathbb{R}$ is a constant. Thus, if O_0 maximizes $\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}^n} q^O(v)_i \big(u_i(v) - u_0(v_0, v)\big) f(v) dv$, then the function U_0 is maximal. $\square$

5.5 Construction of the optimal auction under the new formalism

We now want to describe the optimal auction subject to the assumptions in the theorem above.

We consider an auction format similar to Myerson's format in the section before. In the auction, the seller keeps the object if $\max_{i \in \{1,\dots,n\}} u_i(v) < u_0(v_0, v)$, otherwise the bidder with the highest $u_i(v)$ gets the object. In case of a draw an arbitrary rule may be chosen. By following this format we obtain.

$$\forall i \in \{1, \dots, n\}, 0 < q^O(v)_i \Rightarrow u_0(v_0, v) \leq \max_{j \in \{1,\dots,n\}} u_j(v) = u_i(v)$$

This format maximizes $\sum_{l=1}^{k} \sum_{i \in J_l} \int_{\mathbb{R}^n} q^O(v)_i \big(u_i(v) - u_0(v_0, v)\big) f(x) dx$. Furthermore, for all i between 1 and n, $0 \leq q^O(v)_i$ and $\sum_{l=1}^{k} \sum_{i \in J_l} q^O(v)_i \leq 1$, so the constraints are fulfilled. We can now directly compute the payment functions $O_1, \dots, O_n$.

$$\forall i \in \{1, \dots, n\}, O_i(v) = \begin{cases} u_i(v) & \text{if} \quad q^O(v)_i = 1 \\ 0 & \text{if} \quad q^O(v)_i = 0 \end{cases}$$

Assume now that for all i between 1 and n, $u_i(v) = v_i$ and $u_0(v_0, v) = v_0$. Then, the computed optimal auction is a first price auction where the seller sets a reserve price v_0 and sells the object to the bidder with the highest bid who has to pay his submitted bid.

In our new derivation of the optimal auction we do not use the functions U_i and we do not restrict ourself to the interval $[\alpha, \beta]$. Furthermore, our construction does not require Myerson's regularity assumption in Definition 5.2. By working with the technical lemma we derived for the revenue equivalence theorem we are able to set other requirements for an optimal auction then Myerson.

6 Utility functions and risk

6.1 Relaxations of the RET settings

So far, for our analysis of the revenue equivalence theorem and the optimal auction, we made use of an important assumption regarding the payoff of the bidders participating in the auctions, without directly naming it. Let us state again the expected payoff function we used.

$$a(x, x) - e(x)$$

That is, the expected payoff of a bidder is the difference between the bidder's expected profit and his expected payments. This property is called additive separability.

Definition 6.1 (Additive separability)**.** *A function of two variables $F(x, y)$ is called additively separable, if it can be written as $f(x) + g(y)$ for some single-variable functions $f(x)$ and $g(y)$.*[38]

Translating this defintion into our setting we get that $f(x) := a(x, x)$ and $g(y) = e(y)$. As we will see below this property is not always given and if not, we can not apply the revenue equivalence theorem anymore. In fact, the "additive separability" property is strongly connected to utility functions and utility functions in turn are connected to different risk preferences when we translate the characteristics of utility functions into auction theory.

6.2 Utility functions

In order to consider utility functions of the bidders we introduce the Von Neumann-Morgenstern (VNM) utility functions.

Definition 6.2. *Let $\prec$ be a preference relation over a set of deterministic outcomes Ω. Let p_ω be the probability of the outcome $\omega \in \Omega$, with $\sum_{\omega \in \Omega} p_\omega = 1$. We now introduce four axioms called the axioms of VNM rationality:*

Completeness: *For any ω_1 and $\omega_2 \in \Omega$ there exists $p \in [0, 1]$, such that exactly one of the following holds:*

$$p\omega_1 \prec p\omega_2, \qquad p\omega_1 \succ p\omega_2, \qquad or \qquad p\omega_1 \sim p\omega_2$$

Transitivity: *Let $\omega_1, \omega_2, \omega_3 \in \Omega$ and $p \in [0, 1]$. If $p\omega_1 \preceq p\omega_2$ and $p\omega_2 \preceq p\omega_3$ then*

$$p\omega_1 \preceq p\omega_3$$

[38]See: Bellenot, S. F.: Additivley Separable Functions.

Continuity: *Let $\omega_1, \omega_2, \omega_3 \in \Omega$. If $\omega_1 \preceq \omega_2 \preceq \omega_3$ then there exists $p \in [0,1]$ such that*

$$p\omega_1 + (1-p)\omega_3 \sim \omega_2$$

Independence: *Let $\omega_1, \omega_2 \in \Omega$. If $\omega_1 \preceq \omega_2$ then for any $\omega_3 \in \Omega$ and $p \in (0,1]$:*

$$p\omega_1 + (1-p)\omega_3 \preceq p\omega_2 + (1-p)\omega_3$$[39]

Definition 6.3. *Let $\prec$ be a preference relation over Ω as defined above. Let u be a function of type $\Omega \to \mathbb{R}$ representing $\prec$. The function u has the following characteristics.*

- *$u(0) = 0$.*
- *u is twice continuously differentiable.*
- *u is strictly increasing.*

We will call such a function u a utility function.

Definition 6.4. *The Von Neumann-Morgenstern utility function of u for $\omega \in \Omega$ is defined as follows: Let u be defined as above and let $p \in [0,1]$, then we have for $\omega \in \Omega$:*

$$E(u(pw)) = pu(\omega)$$

The VNM utility function $E(u)$ is therefore the expected value of the utility function u.

6.3 Risk

Definition 6.5. *We distinguish between the following categories of bidders with utility functions u regarding their risk behaviour. A bidder is...*

- *...risk neutral, if $u'' = 0$. In this case u is a linear function of the form $u(x) = ax, a > 0$ and we will write $u \in RN$. Here we can again look back to our setting we used for the revenue equivalence theorem. It becomes clear that we assumed the bidders to be risk neutral, risk neutrality just implies that the expected payoff functions are additively separable.*
- *...(strict) risk averse, if $u'' \leq 0$ ($u'' < 0$). In this case u is a (strict) concave function and we will write $u \in RA$ ($u \in SRA$)*
- *...(strict) risk seeking, if $u'' \geq 0$ ($u'' > 0$). In this case u is a (strict) convex function an we will write $u \in RS$ ($u \in SRS$)*[40]

[39]See: Goyal, V. and Saxena, A.: Von Neumann and Morgenstern Utility Function, http://www.cse.iitd.ernet.in/~rahul/cs905/lecture7/index.html, 2002.

[40]See: Monderer, D. and Tennenholtz, M.: K-Price Auctions: Revenue Inequalities, Utility Equivalence, and Competition in Auction Design, 2003, p. 6.

Definition 6.6. *Two measures for risk averse bidders are the Absolute Risk Aversion Coefficient and the Relative Risk Aversion Coefficient.*
The Absolute Risk Aversion Coefficient of a utility function u at point x is defined as:

$$\lambda_A(x) = -\frac{u''(x)}{u'(x)}$$

A utility function u with constant λ_A for all x is called a CARA utility function.
The Relative Risk Aversion Coefficient of a utility function u at point x is defined as:

$$\lambda_R(x) = x \cdot \lambda_A = -x\frac{u''(x)}{u'(x)}$$

A utility function u with constant λ_R for all x is called a CRRA utility function.[41]

Example 6.7. *An example for a utility function with constant absolute risk aversion is the exponential utility. It is defined as follows:*

$$u_{ex}(x) = 1 - e^{-ax}$$

where a is a positive constant representing the degree of risk aversion. Let us calculate λ_A:

$$\lambda_A = -\frac{u''_{ex}(x)}{u'_{ex}(x)} = -\frac{-a^2e^{-ax}}{ae^{-ax}} = a$$

Example 6.8. *The isoelastic function is an example for a utility function with constant λ_R.*

$$u_{iso}(x) = \begin{cases} \frac{x^{1-r}-1}{1-r} & r \neq 1 \\ \log x & r = 1 \end{cases}$$

with r being a non-negative constant. Calculating the relative risk aversion coefficient we get:

$$\lambda_R(x) = -x\frac{u''_{iso}(x)}{u'_{iso}(x)} = -x\frac{-rx^{-r-1}}{x^{-r}} = -\frac{-rx^{-r}}{x^{-r}} = r$$

Observation 6.9. *Risk averse or risk seeking bidders in the context of auctions can be described as follows:*
Risk averse bidders attach a higher priority to ensure they win the auction compared to a risk neutral bidder. Thus, they are willing to give up parts of their expected utility by bidding higher (compared to the risk neutral bidders) and so increasing their probability to win.[42] *But, an increase of the winning probability is only possible if not all bidders have the same degree of risk aversion. Otherwise all bidders would try to increase their probability to win by bidding higher in the same way. In such a setup, no improvement would be possible for any*

[41] See: Modern Investment Technologies: Multi-period Asset Allocation, 2006, p. 16.
[42] See: Maskin, E and Riley, J.: Auction Theory with Private Values, The American Economic Review, 1985, p. 152.

bidder, since the bidder with the highest value estimation would still win the auction. [43]

John Riley and William Samuelson show in their paper[44] *that the higher the absolut risk aversion coefficient* λ_A *is, the higher the bidders will bid in a first price auction. Furthermore, since the bidding strategy in a second price auction remains unaffected*[45]*, a first price auction should be prefered over a second price auction in case of risk averse bidders as this leads to a higher payoff.*

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[43]See: Kopp, V.: Kontrollierte Auktionen, 2010, pp. 119-120.

[44]Riley, J. and Samuelson, W.: Optimal Auctions, The American Economic Review, Vol 71, No.3., 1981, pp. 381-392.

[45]In a second price auction, the equilibrium strategy for every bidder is to bid an amount equal to his own value. Since the winner of the auction pays the second-highest bid, risk aversion does not affect this dominant bidding strategy.

7 Tailoring auction theory for Eurex Clearing

7.1 Formalization of the problem

In this section we want to characterize the settings of the objectives and plans of the project described in the first section in a mathematically formal way. On the one hand, this will help us to define possible scenarios and a setup for the planned auctions. On the other hand, we will be able to apply possible changes in the future directly to the formalism.
The aim of Eurex Clearing is to find an auction format per liquidation group, so that:

- the whole portfolio will be auctioned off
- there is a minimal consumption of the lines of defense
- the default of the CCP will be prevented

Definition 7.1. *With $C \in \mathbb{N}$ we characterize the number of all participants of Eurex Clearing (i.e. potential auction participants). Let $d \in C$ identify the member who defaults.*

In order to model the portfolio of the defaulted member we use:

Definition 7.2. *Let PF_d be the portfolio of member $d \in C$ at Eurex Clearing:*

$$PF_d = \sum_{i=1}^{M} p_i \times q_i,$$

with $M \in \mathbb{N}$ being the number of all financial instruments traded over the clearing house, p_i are the different instruments and $q_i \in \mathbb{N}_0$ is the number of instruments p_i in the portfolio.

As explained before, the portfolio of the defaulted member will not be auctioned off as a whole, but instead will be split into $l \in \mathbb{N}$ liquidation groups. Then, the different liquidation groups will be hedged properly before the auctions will be performed. Therefore we conclude:

Definition 7.3. *PF_d^H is the hedged portfolio of member d and we split this hedged portfolio into the different liquidation groups:*

$$PF_d^H = \sum_{i=1}^{l} L_{i,d}$$

where $L_{i,d}$ represents liquidation group i of the hedged portfolio PF_d^H and

$$\forall i, j \in \{1, \ldots, l\}, i \neq j \Rightarrow L_{i,d} \cap L_{j,d} = \emptyset,$$

i.e. the sets $L_{i,d}, L_{j,d}$ are disjunct for $i \neq j$.

We also want to consider the possibility of multi-unit auctions (as explained in Section 1.6 Eurex Clearing reserves the right to perform a multi-unit auction instead of running a single-unit auction format). For this possibility of auction formats, the liquidation groups may be split into $t \in \mathbb{N}$ homogenous parts and multi-unit auctions will be performed.

Definition 7.4. *Let $l \in \mathbb{N}$ and for all $i \in \{1, \ldots, l\}$, let $L_{i,d}$ be the liquidation group i of the hedged portfolio PF_d^H. Then we define.*

$$L_{i,d} = t \cdot K_{i,d},$$

where $K_{i,d} \subseteq L_{i,d}$ and $t \in \mathbb{N}$ is the number of homogenous parts of the liquidation group $L_{i,d}$.

In order to model the participants of the different auctions we will use:

Definition 7.5. *Let $N_i = \{1, \ldots, n_i\} \subseteq C \setminus \{d\}$ be the set of participants of the auction of liquidation group $L_{i,d}$. We of course allow that $N_i \subseteq N_j$ for $i \neq j$. Let $k \in \mathbb{N}$. For all $i \in \{1, \ldots, l\}$, let $G_1^{N_i}, \ldots, G_k^{N_i}$ be a partition of N_i.*

7.2 Discussion on the setup

The proposal for the new liquidation procedures foresees that in general, all clearing members and their clients are allowed to participate in the planned auctions. In addition, when entering the auction, all bidders will be able to see the different liquidation groups they will bid on.

Assumption 7.6. *Let $j \in N_i \subseteq C \setminus \{d\}$ be a bidder in the auction of liquidation group $L_{i,d}$. We assume that bidder j takes the following measures into account when determining his valuation for the liquidation group to be auctioned off.*

1. *the market price $x^j_{L_{i,d}} \in \mathbb{R}$ of the liquidation group at the time of the auction*
2. *the impact of the liquidation group on the existing portfolio/the strategy*
3. *additional factors like evaluation of the market orientation of the company in the future*

Assumption 7.7. *There are other factors playing an important role in the auction which do not influence the valuation process of the bidders but may influence their bidding behaviour.*

1. *Considerations regarding the probability of a replenishment of the clearing fund contribution after the liquidation process.*
2. *Considerations regarding the probability of being affected in the allocation process.*

Due to the mark-to-market, the daily evaluation process for all futures and options at the derivatives market from wich the market prices can be derived, we assume that the assumptions on the market price of liquidation group $L_{i,d}$ at the day of the auction are common knowledge and the same for all auction participants N_i, i.e. $\forall j, k \in C \backslash \{d\}, j, k \in N_i \Rightarrow x^j_{L_{i,d}} = x^k_{L_{i,d}}$.

Definition 7.8. *For all $i \in \{1, \dots, l\}$ and for all $j \in N_i$, let $\lambda_i^j \in \mathbb{R}$ be the coefficient characterizing the impact of the liquidation group $L_{i,d}$ on the existing portfolio and the market orientation (and other factors) (Assumption 7.6, 3) of bidder j. We assume the following.*

$$\lambda_i^j \begin{cases} > 1, & \textit{if liquidation group } L_{i,d} \textit{ supports the strategy of the portfolio} \\ = 1, & \textit{if liquidation group } L_{i,d} \textit{ does not influence the strategy of the portfolio} \\ < 1, & \textit{if liquidation group } L_{i,d} \textit{ opposes the strategy of the portfolio} \end{cases}$$

Let us now define the function determining the value of the liquidation group.

Definition 7.9. *For all $i \in \{1, \dots, l\}$ and for all $j \in N_i$, let the function v_i^j be defined as follows.*

$$\begin{array}{llll} v_i^j : & (\mathbb{R})^2 & \to & \mathbb{R} \\ & \lambda_i^j, x^j_{L_{i,d}} & \mapsto & \lambda_i^j \cdot x^j_{L_{i,d}} \end{array}$$

Definition 7.10. *For all $i \in \{1, \dots, l\}$, let $V_i = (V_i^1, \dots, V_i^{n_i})$ be a random vector consisting of n_i real valued random variables.*

Assumption 7.11. *For all $i \in \{1, \dots, l\}$ and for all $j \in N_i$, let f_i^j be the density function of V_i^j. By using the partition $G_1^{N_i}, \dots, G_k^{N_i}$ of the set N_i (see definition 7.5), we introduce the following property for the density functions of the random variables $V_i^1, \dots, V_i^{n_i}$.*

$$\forall s, t \in N_i, f_s = f_t \Rightarrow \exists h \in \{1, \dots, k\}, s, t \in G_h^{N_i}$$

Also, for all h between 1 and k, let the function f^h be defined as follows.

$$\forall j \in N_i, j \in G_h^{N_i} \Rightarrow f^h := f_j$$

This means that we will group the participants of the auction by their density functions.

7.3 Utility functions of the auction participants

Since the potential bidders of the auction only see the liquidation groups after they agree to participate, the decision regarding their utility functions (subject to risk valuation) is taken after they are able to see the liquidation group they will bid on. Not only Eurex Clearing will make every effort to prevent the default of the CCP, it is also in the interest of the Clearing Member that the CCP does not default itself. Losses to the own company would be the consequence for all Clearing Members if the CCP fails. Thus, we will consider different scenarios and analyse independently for each scenario which utility functions are appropriate for the auction participants.

Scenario 1**:** Clearing Member d defaults. But the default of this Clearing Member is not correlated with high market turbulences. The expectation of the participants is that the lines of defense will hold, i.e. the probability of a replenishment of the clearing fund is very low. Considering such a scenario, one can expect that the participants will try to generate profit through the auction process. As described in the previous sections, in the context of auction theory, a risk averse bidder has a higher priority to ensure winning the auction than a risk neutral bidder. Thus, for example in a first price auction, a risk averse bidder will bid higher than a risk neutral bidder. But as explained, in scenario 1 we expect the bidders will try to generate profit. Thus, in the auctions of the liquidation groups one can assume that the bidders do not have utility functions displaying risk aversion. This assumption is also reasonable because the auctions in the liquidation process are not the only possibility for the participants to buy the financial products. So the objects which are sold are not unique, like for example a unique painting. In an auction of a painting it is more reasonable to think about risk averse bidders who are willing to bid higher to increase their probability to win. Let us state these observations in the assumption below.

Assumption 7.12. *In scenario* 1*, for the utility functions u of the bidders in the auctions we will consider functions $u \in RN$ and $u \in RS$ (see Section 6.3).*

Scenario 2**:** Clearing Member d defaults. In scenario 2 we assume that the default of the Clearing Member is highly correlated with market turbulences and the lines of defense of Eurex Clearing may not be sufficient to cover all losses resulting from the default and the succeeding liquidation process. Especially, this case might appear if the prices realised in the auctions, are too low. Thus we assume that in scenario 2 the probability of a replenishment of the clearing fund contribution is relatively higher than in scenario 1. In order to prevent a replenishment or a juniorization, the auction participants thus bid stronger. These observations lead to the following assumption.

Assumption 7.13. *In scenario* 2*, we will consider for the utility functions u of the bidders in the auctions functions $u \in RA$.*

7.4 Constellation of the auction participants

As characterized in assumption 7.11, for all i between 1 and n, we can group the participants N_i of the auction of liquidation group i. In this section we want to discuss a possible setup of such a partition $G_1^{N_i}, \dots, G_k^{N_i}$ of N_i. We want to distinguish between two major groups of auction participants. The aspects under which we will distinguish the auction participants are the clearing member's general strategic focus and as a consequence of that also the time period they take as a basis for their valuation of the portfolio.

Group 1: The clearing members' strategy in general is to benefit from minimal differences between bid and ask prices in the market. The member is not interested in keeping a position very long in his portfolio. Instead, his strategy is to buy a financial product and sell it within a short time period at a profit. For clearing members in this group, we assume a small time period taken as a basis for the valuation of the coefficient λ, by which we characterize the impact of the liquidation group.

Group 2: The clearing member has a certain expectation for the future development of the price of a financial product. Thus, he only buys a financial product when he expects a positive development to be able to sell it at a profit. Thus, for clearing members in this group, we assume a long time period taken as a basis for the valuation of the coefficient λ.

Regarding the question whether the valuation of the bidders is independent from each other, we will distinguish between two cases.

Assumption 7.14. *For all $i \in \{1, \ldots, l\}$, the random variables $V_i^1, \ldots, V_i^{n_i}$ are...*

a) ...mutually independent.

b) ...interdependent.

Regarding the interdependence of the random variables we can assume the model of affiliated random variables Paul Milgrom and Robert Weber introduced for auctions[46]. We just want to characterize the idea behind the model of affiliation: If the realisation of a random variable is a large value then the probability that the other values are also high is greater than the probability that they are small.

7.5 Application of the revenue equivalence theorem

After discussing the setup of an auction for the liquidation groups and possible scenarios and utility functions of the bidders we now want to discuss whether the revenue equivalence theorem we analysed precisely for the single-unit and multi-unit case is applicable in the setup we described or not.
The most important assumption for applying the revenue equivalence theorem is the independence of the random variables $V_i^1, \ldots, V_i^{n_i}$. Thus, in Assumption 7.14, we have to consider possibility a). Remembering the partition of the set $\{1, \ldots, n\}$ we used for the revenue equivalence theorem, we see that by Assumption 7.11 we established an equivalent partition complying with the same rule. If we now take Scenario 1 as a basis for the analysis and assume that

[46] Milgrom, P. and Weber, R.: A Theory of Auctions and Competitive Bidding, Econometrica, Vol. 50, No. 5., 1982, pp. 1089-1122.

all bidders are risk neutral, then all major requirements for the application of the revenue equivalence theorem are fulfilled. But if one of these important requirements is violated, for example when we have risk averse bidders or the valuations (i.e. the random variables) of the bidders are affiliated in the way we indicated above, we can not apply the revenue equivalence theorem anymore and have to consider additional results.

7.6 Conclusion and outlook

The analysis and the modelling of the procedures in case of a default at Eurex Clearing and the discussion regarding a possible setup and scenarios is only a first step in the intensive process of finding the right solution for the challenging problem of Eurex Clearing. Especially, a deep analysis of more complex theorems of auction theory is neccessary in order to be able to give precise recommendations for future procedures. But the detailed analysis of the single-unit and multi-unit versions of the revenue equivalence theorem have shown that even though there exists a lot of work on auction theory, a complete clarification and even generalization is possible for (long-existing) results. Furthermore we were able to show by the revenue equivalence theorem (for example by the partitioning of the bidders) that these generalizations are not only of theoretical interest but have a concrete practical benefit.

By combining the mathematical analysis of auction theory and its theorems with the explicit investigation of the applicability for Eurex Clearing, this thesis can serve as a useful introduction for any institution in search for a good auction format for their specific situation.

Appendix

A Basic probability theory

Definitions in this section are taken from: An introduction to auction theory, by F. Menezes and P. Monteiro and from Introduction to probability theory by C. Geiss and S. Geiss.

Definition A.1 (Sample space)**.** *The sample space Ω is the set of all elementary events ω (of an experiment).*

For example, the sample space of throwing a dice is: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Definition A.2 (σ-Algebra)**.** *Let Ω be a sample space and $\mathcal{P}(\Omega)$ be the set of subsets of Ω. We call $\gamma \subset \mathcal{P}(\Omega)$ a σ-algebra if:*

- $\Omega \in \gamma$
- $\forall A \in \gamma, A^C = \Omega \backslash A \in \gamma$
- *For any family* $(A_n)_{n\in\mathbb{N}} \in \gamma, \bigcup_{n\in\mathbb{N}} A_n \in \gamma$

A pair (Ω, γ), with γ being a σ-algebra on Ω is called a measurable space.

Definition A.3 (Probability measure)**.** *Let (Ω, γ) be a measurable space. We call a function $P : \gamma \to [0, 1]$ a probability measure if:*

- $P(\emptyset) = 0, \quad P(\Omega) = 1$
- *For any family $A_n \in \gamma$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ we have:*

$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

Definition A.4 (Probability space)**.** *We call the triplet (Ω, γ, P) a probability space, with $\gamma \subset \mathcal{P}(\Omega)$ being a σ-algebra and $P : \gamma \to [0, 1]$ a probability measure.*

Definition A.5 (Random variable)**.** *Let $[\Omega, \gamma]$ be a measurable space. The function $X : \Omega \to \mathbb{R}$ is a random variable if for every interval (a, b) of real numbers the set $\{\omega \in \Omega; X(\omega) \in (a, b)\} \in \gamma$. That is, for a random variable the probability of its value being in a given interval is well defined.*

Definition A.6 (Distribution)**.** *A function $F : \mathbb{R} \to [0, 1]$ is a distribution function if:*

- *F is non-decreasing, i.e. $F(x) \leq F(y)$ if $x \leq y$*
- *F is right-continuous, i.e. $F(x+) := \lim_{y\downarrow x} F(y) = F(x)$*
- $F(-\infty) := \lim_{x\to-\infty} F(x) = 0, F(\infty) := \lim_{x\to\infty} F(x) = 1$

Definition A.7. *Given a random variable $X : \Omega \to \mathbb{R}$ its distribution function $F_X : \mathbb{R} \to [0, 1]$ is defined by:*

$$F_X(x) = P([X \le x])$$

Definition A.8 (Density)**.** *The distribution function F has a density if there is a (Riemann integrable) function $f : \mathbb{R} \to \mathbb{R}_+$ such that:*

- *for every x we have: $F(x) = \int_{-\infty}^{x} f(u)du$*
- *$\int_{-\infty}^{\infty} f(x)dx = 1$*

Definition A.9 (Random vector)**.** *A function $X : \Omega \to \mathbb{R}^n$ is a random vector if $\{\omega \in \Omega; X(\omega) \in \prod_{i=1}^{n}(a_i, b_i)\} \subset \mathbb{R}^n \in \gamma$ for every cartesian product of intervals $\prod_{i=1}^{n}(a_i, b_i) \subset \mathbb{R}^n$.*

Definition A.10. *The distribution function $F_X : \mathbb{R}^n \to [0, 1]$ of a random vector X is defined by:*

$$F_X(x_1, \ldots, x_n) = P([X_1 \le x_1, \ldots, X_n \le x_n])$$

The distribution F_X has a density if there exists a Riemann integrable function $f : \mathbb{R}^n \to \mathbb{R}_+$, such that for every $x \in \mathbb{R}^n$:

$$F_X(x) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(y_1, y_2 \ldots y_n) dy_1 dy_2 \ldots dy_n$$

Definition A.11 (Marginal Density)**.** *Let $X = (X_1, \ldots, X_n)$ be a random vector with density f_X. For all i between 1 and n, let f_{X_i} be the marginal density function of the random variable X_i alone. This marginal density can be deduced by integrating the density f_X as follows:*

$$f_{X_i} = \int_{\mathbb{R}^{n-1}} f_X(y_1, \ldots, y_n) dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_n$$

Definition A.12 (Independence)**.** *We call the random variables $X_1, X_2, \ldots, X_m$ independent if:*

$$\begin{aligned} F_X(x) &= P([X_1 \le x_1, \ldots, X_m \le x_m]) \\ &= P([X_1 \le x_1]) \cdots P([X_m \le x_m]) \\ &= F_{X_1}(x_1) \cdots F_{X_m}(x_m) \end{aligned}$$

Corollary A.13. *Suppose that $X_1, \ldots, X_n$ are independent random variables that are identically distributed, i.e. $F := F_{X_1} = \cdots = F_{X_n}$. To find the maximum of the n functions we define $\max\{X_1, \ldots, X_n\}(\omega) = \max\{X_1(\omega), \ldots, X_n(\omega)\}$ for each $\omega \in \Omega$. Now let G be the distribution of the maximum of the X_i's. Then we have:*

$$\begin{aligned} G(t) &= P(\{\omega \in \Omega; \max\{X_1(\omega), \ldots, X_n(\omega)\} \le t\}) \\ &= P([X_i \le t, 1 \le i \le n]) = P(\cap_{i=1}^{n}[X_i \le t]) \\ &= \prod_{i=1}^{n} P([X_i \le t]) = \prod_{i=1}^{n} F(t) = F^n(t) \end{aligned}$$

If F has a density, then G has a density as well:

$$g(t) = G'(t) = n \cdot F^{n-1}(t) \cdot f(t)$$

Definition A.14 (Expectation)**.** *The expectation of a random variable X with distribution function F_X and density f_X is defined by:*

$$E[X] = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

Definition A.15 (Conditional Distribution)**.** *Suppose we have a random vector (X, Y) with distribution F and density $f(\cdot, \cdot)$. The conditional density $f_{X|Y=y}$ is defined by*

$$f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$$

Then, the conditional distribution $F_{X|Y=y}$ is

$$F_{X|Y=y}(x) = \int_{-\infty}^{x} f_{X|Y=y}(t)dt = \int_{-\infty}^{x} \frac{f(t,y)}{\int_{-\infty}^{\infty} f(a,y)da} dt$$

Definition A.16 (Conditional Expectation)**.** *The conditional expectation of a random variable X, given that $X < x$ is*

$$E[X|X < x] = \frac{1}{F_X(x)} \cdot \int_{-\infty}^{x} t \cdot f(t)dt$$

Definition A.17. *Given a random variable X, its density f and a function g, the expected value of $g(X)$ is given by*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Definition A.18. *Given the random variables $X_1, \ldots, X_n$, the sum of the expected values of the n random variables equals the expected value of the sum of the random variables.*

$$\sum_{i=1}^{n} E(X_i) = E(\sum_{i=1}^{n} X_i)$$

This holds, whether the random variables are independent or not.

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